Difference Equations
to
Differential Equations

Section 1.1

Calculus: Areas And Tangents

The study of calculus begins with questions about change. What happens to the velocity of a swinging pendulum as its position changes? What happens to the position of a planet as time changes? What happens to a population of owls as its rate of reproduction changes? Mathematically, one is interested in learning to what extent changes in one quantity affect the value of another related quantity. Through the study of the way in which quantities change we are able to understand more deeply the relationships between the quantities themselves. For example, changing the angle of elevation of a projectile affects the distance it will travel; by considering the effect of a change in angle on distance, we are able to determine, for example, the angle which will maximize the distance.

Related to questions of change are problems of approximation. If we desire to approximate a quantity which cannot be computed directly (for example, the area of some planar region), we may develop a technique for approximating its value. The accuracy of our technique will depend on how many computations we are willing to make; calculus may then be used to answer questions about the relationship between the accuracy of the approximation and the number of calculations used. If we double the number of computations, how much do we gain in accuracy? As we increase the number of computations, do the approximations approach some limiting value? And if so, can we use our approximating method to arrive at an exact answer? Note that once again we are asking questions about the effects of change.

Two fundamental concepts for studying change are sequences and limits of sequences. For our purposes, a sequence is nothing more than a list of numbers. For example,

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

might represent the beginning of a sequence, where the ellipsis indicates that the list is to continue on indefinitely in some pattern. For example, the 5th term in this sequence might be

$$\frac{1}{16} = \frac{1}{2^4},$$

the 8th term

$$\frac{1}{128} = \frac{1}{2^7},$$

and, in general, the nth term

$$\frac{1}{2^{n-1}}$$

where $n = 1, 2, 3, \ldots$. Notice that the sequence is completely specified only when we have given the general form of a term in the sequence. Also note that this list of numbers is approaching 0, which we would call the limit of the sequence. In the next section of this chapter we will consider in some detail the basic question of determining the limit of a sequence.

The following two examples consider these ideas in the context of the two fundamental problems of calculus. The first of these is to determine the area of a region in the plane; the other is to find the line tangent to a curve at a given point on the curve. As the course progresses, we will find that general methods for solving these two problems are at the heart of the techniques used in calculus. Moreover, we will see that these two problems are, surprisingly, closely related, with the area problem actually being the inverse of the tangent problem. This intimate connection was one of the great discoveries of Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716), although anticipated by Newton's teacher Isaac Barrow (1630-1677).

Example Suppose we wish to find the area inside a circle of radius one centered at the origin. Of course, we have all learned that the answer is π. But why? Indeed, what does it mean to find the area of a disk?

Area is best defined for polygons, regions in the plane with line segments for sides. One can start by defining the area of a 1×1 square to be one unit. The area of any other polygonal figure is then determined by how many squares may be fit into it, with suitable cutting as necessary. For example, it is seen that the area of a rectangle with base of length b and height a should be ab. Since a parallelogram with base of length b and height a may be cut and pasted onto a rectangle of length b and height a (see Problem 1), it follows that the area of such a parallelogram is also ab. As a triangle with height a and a base of length b is one-half of a parallelogram of height a and base length b (see Problem 2), it easily follows that the area of such a triangle is $\frac{1}{2}ab$. The area of any other polygon can be calculated, at least in theory, by decomposing it into a suitable number of triangles. However, a circle does not have straight sides and so may not be handled so easily. Hence we resort to approximations.

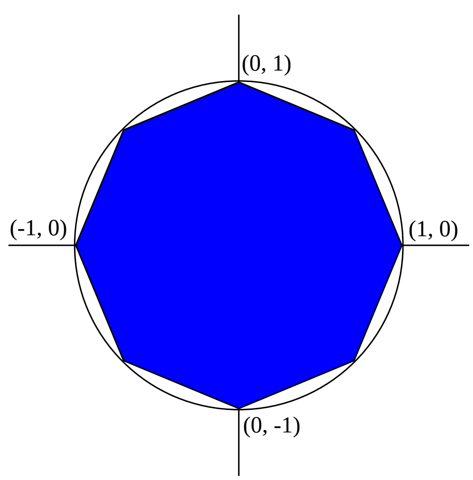

Figure 1.1.1 A regular octagon inscribed in a unit circle

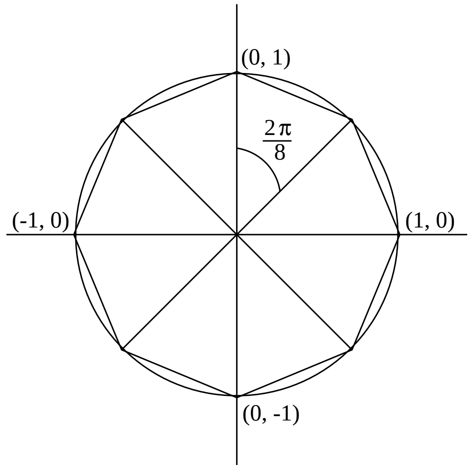

Figure 1.1.2 Decomposition of a regular octagon into eight isosceles triangles

Let P_n be a regular n-sided polygon inscribed in the unit circle centered at the origin and let A_n be the area of P_n. For example, Figure 1.1.1 shows P_8 inscribed in the unit circle. We may decompose P_n into n congruent isosceles triangles by drawing line segments from the center of the circle to the vertices of the polygon, as shown in Figure 1.1.2 for P_8. For each of these triangles, the angle with vertex at the center of the circle has measure $\frac{360}{n}$ degrees, or $\frac{2\pi}{n}$ radians, where π represents the ratio of the circumference of a circle to its diameter. Hence, since the equal sides of each of the triangles are of length one, each triangle has a height of

$$h_n = \cos\left(\frac{\pi}{n}\right)$$

and a base of length

$$b_n = 2\sin\left(\frac{\pi}{n}\right)$$

(see Problem 3). Thus the area of a single triangle is given by

$$\frac{1}{2}b_n h_n = \cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{n}\right),$$

where we have used the fact that

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

for any angle α. Multiplying by n, we see that the area of P_n is

$$A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right).$$

We now have a sequence of numbers, A_1, A_2, A_3, $\ldots$, each number in the sequence being an approximation to the area of the circle. Moreover, although not entirely obvious, each term in the sequence is a better approximation than its predecessor since the corresponding regular polygon more closely approximates the circle. For example, to five decimal places we have

$$\begin{aligned}
A_3 &= 1.29904,\\
A_4 &= 2.00000,\\
A_5 &= 2.37764,\\
A_6 &= 2.59808,\\
A_7 &= 2.73641,\\
A_8 &= 2.82843,\\
A_9 &= 2.89254,\\
A_{10} &= 2.93893,\\
A_{11} &= 2.97352,
\end{aligned}$$

and

$$A_{12} = 3.00000.$$

Continuing in this manner, we find $A_{20} = 3.09017$, $A_{50} = 3.13333$, and $A_{100} = 3.13953$. As we would expect, the sequence is increasing and appears to be approaching π. Indeed, if we take a polygon with 1644 sides, we have $A_{1644} = 3.14159$, which is π to five decimal places.

Alternatively, instead of defining π to be the ratio of the circumference of a circle to its diameter, we could define it to be the area of a circle of radius one. That is, we could define π to be the limiting value of the sequence A_n. Symbolically, we express this by writing

$$\pi = \lim_{n\to\infty} A_n.$$

In that case, let B be the area of a circle of radius r and let B_n be the area of a regular n-sided polygon Q_n inscribed in the circle. If we decompose Q_n into n isosceles triangles in the same manner as P_n above, then each triangle in this decomposition is similar to any one of the triangles in the decomposition of P_n. Since the ratios of the lengths of corresponding sides of similar triangles must all be the same, the sides of a triangle in the decomposition of Q_n must be r times the length of the corresponding sides of any triangle in the decomposition of P_n. Hence each of the triangles in the decomposition of Q_n must have a base of length rb_n and a height of rh_n, where h_n is the height and b_n is the length of the base of one of the isosceles triangles in the decomposition of P_n. Thus the area of one of the triangles in the decomposition of Q_n into isosceles triangles will be

$$\frac{1}{2}(rb_n)(rh_n) = \frac{1}{2}r^2 b_n h_n,$$

from which it follows that

$$B_n = \frac{n}{2}r^2 b_n h_n = r^2\left(\frac{n}{2}b_n h_n\right) = r^2 A_n.$$

Since r is a fixed constant, we would then expect that, in the limit as the number of sides grows toward infinity,

$$B = \lim_{n\to\infty} B_n = \lim_{r^2 A_n} = r^2 \lim_{n\to\infty} A_n = \pi r^2.$$

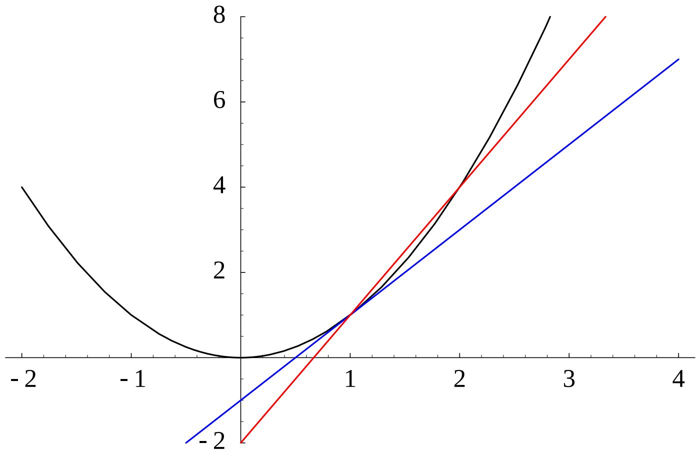

Figure 1.1.3 Parabola $y = x^2$ with tangent line (blue) and a secant line (red)

Hence we arrive at the famous formula for the area of a circle of radius r, in which the constant π has been defined to be the area of a circle of radius one.

Example In this example we wish to find the line tangent to the curve $y = x^2$, a parabola, at the point $(1, 1)$. This problem may not at first seem as useful as that of finding the area of a planar region, but we shall find that the ideas behind the solution have many applications, and are, ultimately, important in the solution of the area problem as well.

First there is the question of exactly what is a tangent line. At the present it will be sufficient to leave the notion at an intuitive level: a tangent line is a line which just touches a given curve at a point, giving a close approximation between curve and line. In Chapter 3, we will see that a line ℓ is tangent to a curve C at a point P on C if ℓ passes through P and, in a sense that we will make precise at that time, gives a better approximation to C for points close to P than any other line.

Now let C be the curve with equation $y = x^2$, let $P = (1, 1)$, and let ℓ be the line tangent to C at P. Since ℓ passes through P, in order to find the equation of ℓ we need only find its slope m. Unfortunately, to find m in the standard way we need to know two points on ℓ, and we know only one, namely P. Hence we will again have to resort to approximations. For example, the line through the points $(1, 1)$ and $(2, 4)$ is not ℓ (it is a secant line, rather than a tangent line), but since it intersects C at P and at another point which is close to P, its slope should approximate m (see Figure 1.1.3). Namely, we have

$$m \approx \frac{4-1}{2-1} = 3.$$

Since $(\frac{3}{2}, \frac{9}{4})$ is on C and is closer to P than $(2, 4)$, a better approximation is given by the slope of the line passing through $(1, 1)$ and $(\frac{3}{2}, \frac{9}{4})$, that is,

$$m \approx \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{\frac{5}{4}}{\frac{1}{2}} = \frac{5}{2}.$$

More generally, let n be a positive integer and let m_n be the slope of the line through the points

$$\left(1+\frac{1}{n}, \left(1+\frac{1}{n}\right)^2\right)$$

and P. For example, we have just seen that $m_1 = 3$ and $m_2 = \frac{5}{2}$. Now, in general,

$$\begin{aligned} m_n &= \frac{\left(1+\frac{1}{n}\right)^2 - 1}{\left(1+\frac{1}{n}\right) - 1} \\ &= \frac{1 + \frac{2}{n} + \frac{1}{n^2} - 1}{\frac{1}{n}} \\ &= n\left(\frac{2}{n} + \frac{1}{n^2}\right) \\ &= 2 + \frac{1}{n} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Hence

$$\begin{aligned} m_3 &= 2 + \frac{1}{3} = \frac{7}{3}, \\ m_4 &= 2 + \frac{1}{4} = \frac{9}{4}, \\ m_5 &= 2 + \frac{1}{5} = \frac{11}{5}, \end{aligned}$$

and so on. Moreover, as n increases, $\frac{1}{n}$ decreases toward 0, and so we would expect that as n increases, m_n decreases toward 2. At the same time, as n increases m_n more closely approximates m. Thus we should have

$$m = \lim_{n\to\infty} m_n = \lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = 2.$$

That is, the slope of the line tangent to C at P is 2. Then the tangent line ℓ has equation

$$y - 1 = 2(x - 1),$$

or

$$y = 2x - 1.$$

Here we have used the fact that the equation of a line with slope m and passing through the point (a, b) is given by

$$y - b = m(x - a).$$

The rest of this chapter will be concerned with the study of sequences and their limits. The next section will consider the basic definitions and computational techniques, while the remaining sections will discuss some applications. We will return to the problem of finding tangent lines in Chapter 3 and the problem of computing areas in Chapter 4.

Problems

1. Use Figure 1.1.4 to verify that a parallelogram with height a and base of length b has area ab.

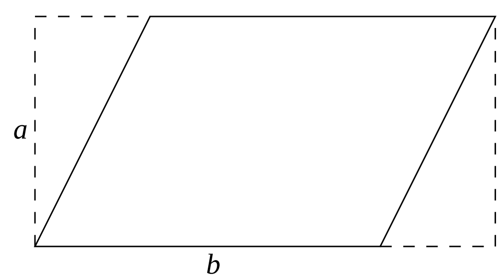

Figure 1.1.4 A parallelogram

2. Explain how any triangle is one-half of a parallelogram, and use this to verify the formula for the area of a triangle.

3. Use Figure 1.1.5 to verify the formulas given for the height and base of one of the isosceles triangles in the decomposition of P_n.

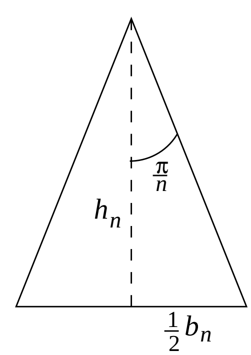

Figure 1.1.5 An isosceles triangle from the decomposition of P_n

4. Try the procedure of the tangent example to find the equation of the line tangent to the following curves at the indicated point.

   (a) $y = 2x^2$ at $(1, 2)$ (b) $y = x^2 + 1$ at $(1, 2)$

   (c) $y = x^3$ at $(1, 1)$ (d) $y = x^2$ at $(2, 4)$

5. For the area example, find the number of sides necessary for the area of the inscribed polygon to approximate π to 6, 7, 8, 9, and 10 digits after the decimal point.

6. For the tangent example, how large would n have to be in order for $|m_n - 2|$ to be less than 0.005?

7. For the tangent example, let p be the smallest positive integer such that $|m_p - 2| < 0.01$.
   (a) What is p?
   (b) What can you say about $|m_n - 2|$ for values of n greater than p?

8. For each of the following sequences $\{a_n\}$, compute a_{10}, a_{20}, a_{100}, a_{500}, and a_{1000}.
   (a) $a_n = n \sin\left(\frac{1}{n}\right)$
   (b) $a_n = \left(1 + \frac{1}{n}\right)^n$
   (c) $a_n = \frac{10^n}{n!}$, where $n! = n(n-1)(n-2)\cdots(2)(1)$

9. As we saw in the area example, there is more than one way to define the number π. For example, we can define it either as the area of a circle of unit radius or as the ratio of the circumference of a circle to its diameter (of course, if the latter approach is taken, one has to show that this ratio is the same for every circle). Suppose we define π as the area of a circle of unit radius. Consider a circle with radius r, diameter d, circumference C, and area A. Then we have seen that $A = \pi r^2$. The following steps show that we also have $\pi = \frac{C}{d}$.
   (a) Let P_n be a regular n-sided polygon inscribed in the circle. Let s be the length of a side of P_n. By dividing P_n into n equal isosceles triangles as we did in the area example, argue that
   $$A \approx \frac{nrs}{2}.$$
   (b) Can you see why as n goes to infinity, ns approaches C?
   (c) Now can you see why
   $$A = \lim_{n\to\infty} \frac{nrs}{2} = \frac{rC}{2}?$$
   (d) Use the result in part (c) to show that
   $$\pi = \frac{C}{d}.$$

10. You may find an interesting discussion of techniques for computing areas and volumes up to the time of Archimedes (287-212 B.C.) in the first two chapters of *The Historical Development of Calculus* by C. H. Edwards (Springer-Verlag New York Inc., 1979). In particular, there is a discussion on pages 31-35 of Archimedes' proof that the two definitions of π mentioned in the area example yield the same number.