Difference Equations
to
Differential Equations

Section 2.1

Functions And Their Graphs

Since functions are the basic building blocks out of which mathematicians construct models of the physical world, it is essential that any student of mathematics have a firm grasp of the concept. In particular, one must be careful to distinguish between a given function and a notational or graphical representation for it. A function is a type of relationship, a mental concept that cannot be seen or touched. Although pictures and symbolic representations of a function are extremely important in understanding its behavior, the student must always keep in mind the distinction between the function itself and its representations.

Modern methods for giving a formal definition of a function, developed in the latter part of the 19th century, are based on set-theoretic ideas. We will not go into the details necessary to make such a precise definition, but rather aim at an intuitive understanding of the basic concept. For us, a function is a special type of relationship between two quantities. We often think of this relationship to be one of dependence. That is, if the value of one quantity, say y, is determined by the value of another quantity, say x, then we say that y is a function of x. For example, if x represents the height from which a certain rock is dropped and y represents the velocity with which the rock strikes the ground, then the value of y will depend on the value of x and we say that velocity y is a function of height x. Note here that if y is the terminal velocity of the object, then there are many different values of x which yield the same value of y, namely, any value of x which gives the object sufficient time to reach its terminal velocity before striking the ground. On the other hand, for a given value of x, there is only one related value of y. It is this latter property that makes the relationship between height and impact velocity a function. For any quantities represented by y and x, in order to say that y is a function of x we require that every value of x be related to exactly one value of y. Such a relationship often arises through some physical dependency, a cause creating a deterministic effect, but the definition does not require such a link between the quantities in question. A number of examples should help clarify this concept.

Example Sequences are example of functions. That is, if $\{x_n\}$ is a sequence with $n = 1, 2, 3, \ldots$, then every value of n determines exactly one value x_n. For example, the area of a regular polygon inscribed in a unit circle is a function of the number of sides. Also, a difference equation, such as

$$x_{n+1} = 1.02x_n,$$

$n = 0, 1, 2, \ldots$, makes x_n a function of n. For example, the size of a certain population of owls will be a function of the number of years from some starting date.

Example The area of a circle is a function of the radius of the circle.

Example The distance of the earth from the sun is a function of the time of year.

Example The temperature at a certain fixed point in space is a function of time.

In mathematical terminology, if y is a function of x, then we call x the *independent variable* and y the *dependent variable*. Also, the *domain* of this function is the set of permissible values for x and the *range* is the set of all values of y which correspond to some value of x.

Example Recall that the nth term of the sequence which gives the area of a regular n-sided polygon inscribed in a unit circle is

$$A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right),$$

$n = 3, 4, 5, \ldots$ The domain of this function is the set of integers $\{3, 4, 5, \ldots\}$. The range can be specified only by saying that it is the set of numbers

$$\left\{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \;\middle|\; n = 3, 4, 5, \ldots\right\}.$$

Before proceeding further, we should recall the notation for intervals of real numbers. Given any real numbers a and b, we have

$$(a, b) = \{x \mid a < x < b\}, \tag{2.1.1}$$

$$(a, b] = \{x \mid a < x \leq b\}, \tag{2.1.2}$$

$$[a, b) = \{x \mid a \leq x < b\}, \tag{2.1.3}$$

$$[a, b] = \{x \mid a \leq x \leq b\}, \tag{2.1.4}$$

$$(a, \infty) = \{x \mid x > a\}, \tag{2.1.5}$$

$$[a, \infty) = \{x \mid x \geq a\}, \tag{2.1.6}$$

$$(-\infty, b) = \{x \mid x < b\}, \tag{2.1.7}$$

and

$$(-\infty, b] = \{x \mid x \leq b\}. \tag{2.1.8}$$

Moreover, we call intervals of the form (2.1.1), (2.1.5), and (2.1.7) *open intervals* and intervals of the form (2.1.4), (2.1.6), and (2.1.8) *closed intervals*.

Example If we let d specify the distance from the sun to the earth and t specify the time of year, then the function that relates d and t has domain

$$\{t \mid 0 \leq t \leq 8760\} = [0, 8760],$$

where t is specified in hours, and range

$$\{d \mid 91.4 \leq d \leq 94.6\} = [91.4, 94.6],$$

where d is specified in millions of miles.

Before learning much about a specific function, a mathematician must represent the function in some concrete form. This can be done in many ways. For example, we might construct a table of values for the function. Such a table might have two rows, one for values of the independent variable and one for the corresponding values of the dependent variable. For example, if T is the temperature, in degrees Fahrenheit, at the Kalispell airport weather station at time t, measured in hours past midnight, on August 3, 1999, then our table might look like the following:

Time (t)	0	1	2	3	4	5	6	7	8	9
Temperature (T)	68	66	64	62	61	59	60	64	68	70
Time (t)	10	11	12	13	14	15	16	17	18	19
Temperature (T)	74	76	78	80	84	84	82	81	79	77
Time (t)	20	21	22	23	24					
Temperature (T)	74	70	69	67	66					

In other words, this table provides a complete listing of the values of the dependent variable T which correspond to each value of the independent variable t.

Of course, if the domain of the function contains a large number of points, it might not be practical to represent the function using a table. Indeed, most of the functions which we will consider in this course have an infinite number of points in their domain, rendering complete representations using tables impossible. Moreover, even with a limited amount of data, it is hard to understand much about the underlying function by looking at a table. One alternative to a tabular representation of a function is a graphical representation. If y is a function of x, the graph of this function is the set of all points in the Cartesian plane with coordinates (x, y). If the domain of the function has only a finite number of points, as in the preceding example, then its graph is just a set of points in the plane, as we see in Figure 2.1.1. However, if we were able to plot this function for all values of t between 0 and 24, then its graph would become a curve passing through the points given by the table. With the given data, we could approximate this curve by plotting the given points and then connecting successive points by straight lines, as in Figure 2.1.2. In either form, the graph gives a good pictorial representation of the function. From this picture, we can easily identify such things as the high and low temperatures for the day, as well as the time at which they occurred, or the time of day when the temperature was changing most rapidly.

It would be hard to overestimate the importance of graphs in studying functions; we will in fact spend much time in this course considering graphs. However, the most concise, and at the same time most complete, representation for a function is a formula which expresses the values of the dependent variable in terms of values of the independent variable. For a given function it may not be possible to find such a formula. For example, the function which gives the temperature at the Kalispell airport for any given time during the day of August 3, 1999, is not expressible by a formula; the only way we can compute values for this function is to record the temperatures as they occur. On the other hand, for a circle of radius r and area A, the formula $A = \pi r^2$ gives us an explicit means for computing values of the dependent variable A for any given value of the independent variable r. The

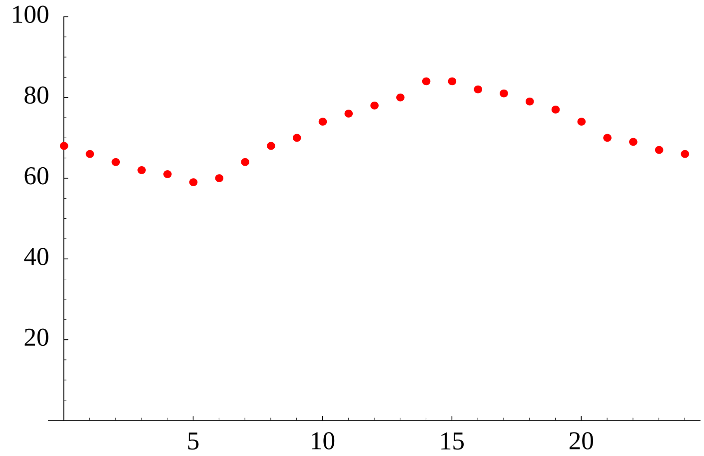

Figure 2.1.1 Plot of temperature data for Kalispell

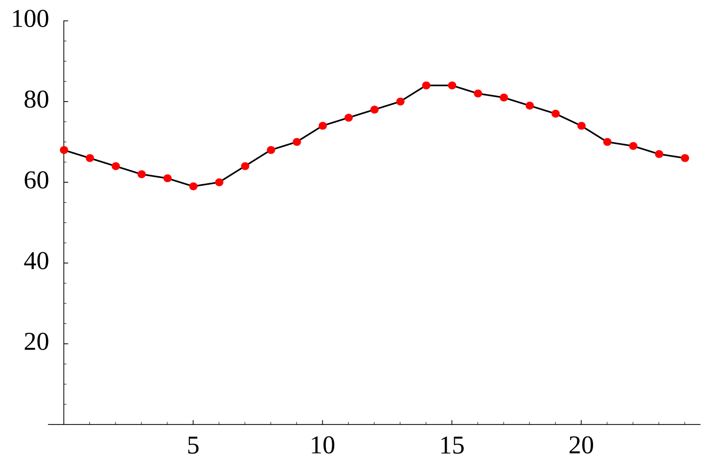

Figure 2.1.2 Plot of temperature date for Kalispell with lines connecting data points

existence of a formula for a function enables us to perform mathematical computations which, at best, could only be approximated otherwise. At the same time, it is important to remember that a function is an abstract object; it is not itself a formula or a number or a graph, but a relationship which exists between quantities specified by numbers. We need to keep this in mind, even as we proceed to work more and more with functions through their representations using formulas and graphs.

Example If V represents the volume and r the radius of a sphere, then V is a function of r and the formula

$$V = \frac{4}{3}\pi r^3$$

expresses this relationship. Note that the domain of this function is the open interval $(0, \infty)$, even though negative values of r can be substituted into the formula without any problems. This emphasizes that the function is determined by the underlying relationship between V and r. Here we also have the range equal to $(0, \infty)$.

Example Suppose the quantity y is related to the quantity x by the formula

$$y = \frac{1}{\sqrt{1 - x^2}}.$$

Since, by convention, the square root notation refers to the positive square root of a given number, this relationship makes y a function of x. If we are given no further information about this function, then we should ascribe to it the largest possible domain and range. In this case, the domain is the interval $(-1, 1)$ (that is, values of x for which $1 - x^2$ is positive) and the range is $[1, \infty)$ (that is, the possible results from dividing 1 by numbers in the interval $(0, 1]$).

At this point, we have used notation for the dependent and independent variables of a function, but not for the function itself. As with variables, it is common to use letters to designate functions. For example, we frequently use f to denote a function, in which case f stands for the function itself, a relationship, while expressions like $f(x)$, $f(2)$, and $f(s)$ denote particular values of the function. That is $f(x)$, $f(2)$, and $f(s)$ represent values of the dependent variable which correspond to the values x, 2, and s, respectively, of the independent variable.

Example The expression

$$f(x) = \frac{1}{x^2}$$

tells us that f represents a function which associates the value

$$\frac{1}{x^2}$$

to a given value x of the independent variable. Hence, for example,

$$f(2) = \frac{1}{4},$$

$$f(-1) = 1,$$

$$f(s) = \frac{1}{s^2},$$

and

$$f(z+1) = \frac{1}{(z+1)^2}.$$

Note that the domain of f is $\{x \mid x \neq 0\}$. That is, f is defined for every real number except 0. The range of f is $(0, \infty)$.

Example Suppose S is the function which gives the temperature at the Kalispell airport on August 3, 1990. If we measure time in of hours since midnight, then we know, for example, that $S(2) = 64$ and $S(19) = 77$. However, if we let t represent the independent variable for this function, namely, the number of hours since midnight, then we do not have a general formula to express $S(t)$. For example, we cannot compute $S(7.5)$ or $S(3.2)$, let alone even consider what $S(\pi)$ might be.

It often happens that the output from one function is used as input for another function. For example, suppose a pebble is dropped in a pond and the resulting circular wave has a radius of $20t$ centimeters after t seconds. Then if r is the radius of the wave and A is the area inside the wave, we have $r = 20t$ and $A = \pi r^2$. But the area inside the wave is also a function of time, which may be expressed as

$$A = \pi(20t)^2 = 400\pi t^2.$$

That is, the area of the circle is a function of the radius, which in turn is a function of time. The function that we arrive at, namely, A as a function of t, is called the *composition* of the two original functions. In the notation which uses letters to denote functions, we have the following definition.

Definition If f and g are two functions, then the composition of f and g is the function $f \circ g$ whose value at x is given by

$$f \circ g(x) = f(g(x)). \tag{2.1.9}$$

Example If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, then

$$f \circ g(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

and

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = x + 1.$$

Note that $f \circ g$ and $g \circ f$, as in this example, are not usually the same function.

Classes of functions

The simplest type of functions are those which involve only multiplication and addition. In particular, functions of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{2.1.10}$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a nonnegative integer, are called *polynomials*. If $a_n \neq 0$, the *degree* of the polynomial is n. For example,

$$q(x) = 3x^2 - 13x + 3,$$

$$f(t) = 21t^{34} + 18t^2 - \pi,$$

and

$$g(s) = \frac{1}{2}s^3 + s + 12 - 4s^5$$

are polynomials, of degrees 2, 34, and 5, respectively, whereas

$$h(x) = \frac{3}{x}$$

is not a polynomial. In a sense polynomials are the building blocks for a large family of important functions in calculus. In this regard, one of the major goals of this text is to show how polynomials may be used to approximate more complicated functions.

Functions which may be written in the form of a polynomial divided by a polynomial are called *rational functions*. For example,

$$f(x) = \frac{3x^2 - 4x + 1}{x^4 + 1}$$

and

$$g(s) = \frac{12}{s} + \frac{1}{s^2 - 3s + 1}$$

are both rational functions, the latter because it may be rewritten as a polynomial divided by a polynomial if all the terms are put over the common denominator $s(s^2 - 3s + 1)$.

The function $f(x) = \sqrt{x}$ is neither a polynomial nor a rational function because x is raised to a power which is not an integer. Functions which permit addition, multiplication, division, and rational numbers for powers are called *algebraic functions*. Thus, for example,

$$g(t) = t^{\frac{3}{2}} + 2t^2 - 3$$

is neither a polynomial nor a rational function, but is an algebraic function. Similarly,

$$h(s) = \sqrt{s^2 + 3s + 2}$$

is an algebraic function. We should note that every polynomial is also a rational function and every rational function is also an algebraic function.

Functions which are not algebraic are called *transcendental*. The trigonometric functions are examples of transcendental functions. We shall discuss them in detail in the next section.

Graphs of functions

We are now in a position to say more about the graphs of functions. With the notation we have now, the graph of a function f is the set of all points $(x, f(x))$ in the plane, where x is in the domain of f. For example, you should recall from previous work that the graph of $y = x^2$ is a parabola opening about the x-axis with its vertex at $(0, 0)$. Also, you should recall the shapes of the graphs of such functions as $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$. See Figures 2.1.3, and 2.1.4.

Moreover, given a function f and a constant c, you may recall that the graph of $y = f(x) + c$ is the graph of f shifted c units vertically (upward if $c > 0$ and downward if $c < 0$), the graph of $y = f(x - c)$ is the graph of f shifted c units horizontally (to the right

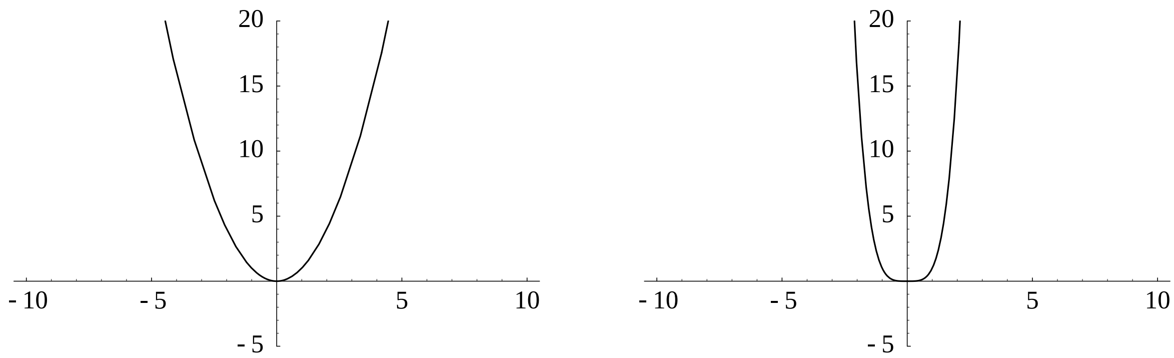

Figure 2.1.3 Graphs of $y = x^2$ and $y = x^4$

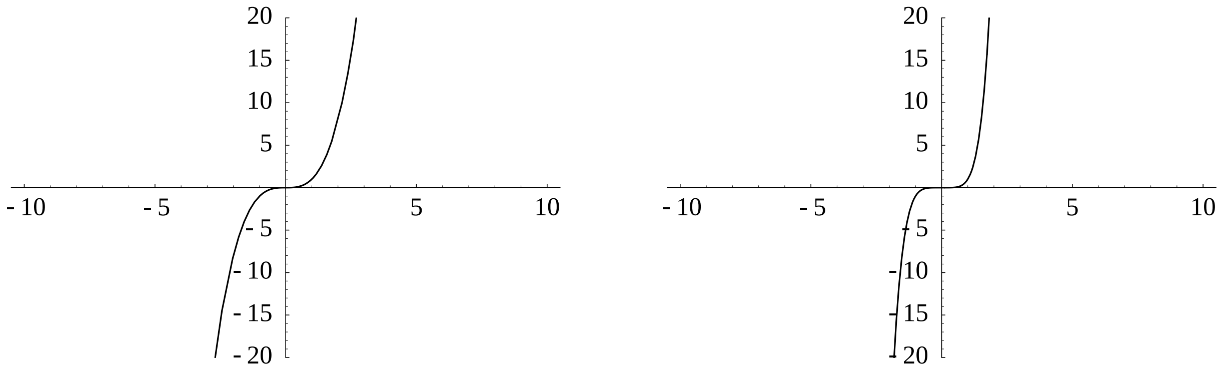

Figure 2.1.4 Graphs of $y = x^3$ and $y = x^5$

if $c > 0$ and to the left if $c < 0$), and the graph of $y = -f(x)$ is the graph of f reflected about the x-axis. Hence, for example, the graph of

$$y = x^2 - 3$$

is a parabola, opening upward about the x-axis, with its vertex at $(0, -3)$; the graph of

$$y = (x + 2)^2 - 3$$

is a parabola, opening upward about the line $x = -2$, with vertex at $(-2, -3)$; and the graph of

$$y = -(x + 2)^2 + 3$$

is a parabola, opening downward about the line $x = -2$, with vertex at $(-2, 3)$. See Figure 2.1.5.

Drawing graphs of functions whose basic shapes are not already known to us can be a difficult problem. If the domain of a function f is finite, then drawing its graph is only a matter of plotting some points in the plane. However, most of the functions we will encounter in this course will have domains containing an infinite number of points; drawing the graphs of such functions requires much more than plotting a few points. The

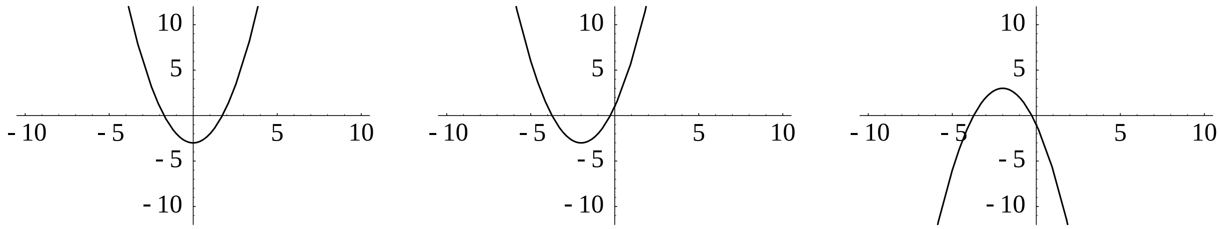

Figure 2.1.5 Graphs of $y = x^2 - 3$, $y = (x+2)^2 - 3$, and $y = -(x+2)^2 + 3$

problem is that no matter how many points we plot, we still do not know how the function is behaving at the other points. For example, if we want to graph a function f on the interval $[0, 1]$, we might first plot the points

$$(0, f(0)), (0.1, f(0.1)), (0.2, f(0.2)), \ldots, (1.0, f(1.0)).$$

Next, to guess at the behavior of the function between the plotted points, we might join successive points by straight lines. Of course, this will only give us an approximation to the true curve, the accuracy of which will depend on the actual behavior of the curve between the plotted points, something about which we frequently have very little information. This is similar to the problem we had with plotting the graph of a temperature function earlier. However, here we can get help if we have a formula for f; for in that case we can try plotting more points, say

$$(0, f(0)), (0.05, f(0.05)), (0.10, f(0.10)), \ldots, (1.00, f(1.00)),$$

or

$$(0, f(0)), (0.01, f(0.01)), (0.02, f(0.02)), \ldots, (1.00, f(1.00)).$$

If the graph of f is a reasonably smooth curve, we will be able to approximate it as well as we like by plotting a sufficient number of points. This raises two questions: How do we know that we have plotted a sufficient number of points? And, given that a sufficient number of points will most likely be a large number, how do we actually plot them all? Of course, the latter question is answered by using a computer. In fact, this approach to graphing a function is unreasonable without access to a computer, or at least a calculator. Computers also provide help in answering the first question. We start by plotting a reasonable number of points, say 100 or so. If we have reason to doubt the accuracy of the resulting graph, perhaps because the curve is not as smooth as we expected it to be, we can double the number of points and plot it again. Because any computer, and in fact many calculators, do this type of work rapidly, it is reasonable to plot the same function several times until we are comfortable with the picture. In Section 3.9 we will learn how to use some of the techniques of calculus to better understand the geometry of the graph of a function. This will help us identify whether or not the output from a computer is an accurate depiction of the graph.

Example Figure 2.1.6 compares the results of plotting the points

$$(0, f(0)), (0.1, f(0.1)), (0.2, f(0.2)), \ldots, (1.0, f(1.0))$$

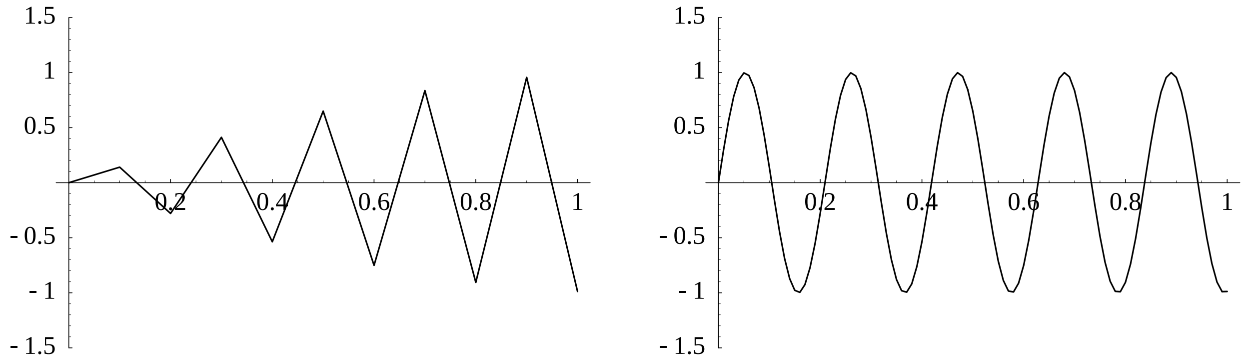

Figure 2.1.6 Plot of $y = \sin(30x)$, using first 11 points and then 101 points

with plotting the points

$$(0, f(0)), (0.01, f(0.01)), (0.02, f(0.02)), \ldots, (1.00, f(1.00))$$

(joining successive points with straight lines) for the function $f(x) = \sin(30x)$. Clearly, the second plot a dramatic improvement over the first.

When using a computer software package or a calculator to graph a function, there are a couple of issues you should keep in mind. First, as we have just noted, plotting a sufficient number of points is crucial to obtaining a good approximation of the graph. Some programs will ask you to specify the number of points to plot, others will plot a predetermined number of points, and still others will determine the number of points to plot based on an estimate of the number of points necessary to provide an accurate picture for the particular function. In the latter two cases it should still be possible to override the program's decision and specify your own choice for the number of points to plot. If plotting more points significantly changes the look of the graph, then you should be wary of the original plot and consider plotting even more points. Second, the computer will plot the function in a rectangle, called a window. The horizontal scale for this window will be the interval over which you want to graph the function. The vertical scale may be chosen by you or by the computer program. If possible, it is usually best first to let the program choose the vertical scale for the window and then to adjust it as necessary to provide a good picture of the graph. If the vertical scale is too small, you may miss part of the graph; if the vertical scale is too large, the interesting features of the graph may be too small to be visible. For example, Figure 2.1.7 shows the graph of $y = \sin(x)$ on the interval $[-4, 4]$, first with the vertical window scale being the interval $[-0.5, -0.5]$ and next with the vertical scale changed to $[-20, 20]$. Certainly, a vertical window scale on the order of $[-1.5, 1.5]$, as shown in Figure 2.1.8, is a more appropriate choice for this graph.

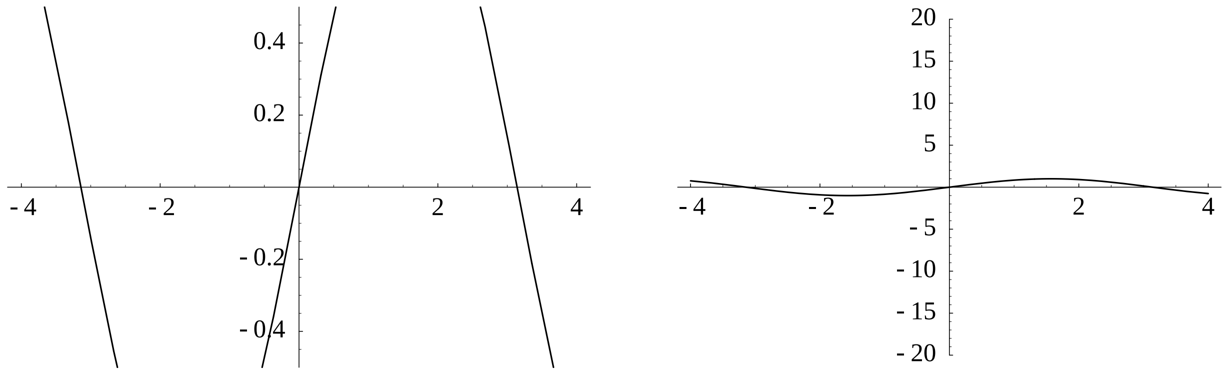

Figure 2.1.7 Graph of $y = \sin(x)$, first with vertical window $[-0.5, 0.5]$ and then $[-20, 20]$

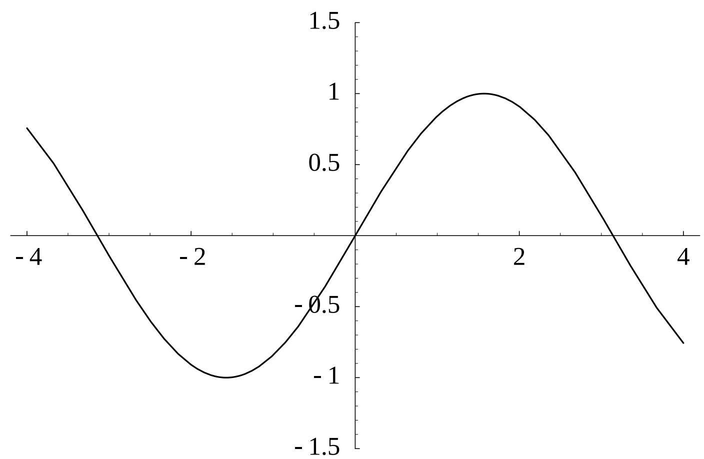

Figure 2.1.8 Graph of $y = \sin(x)$ with vertical window $[-1.5, 1.5]$

Problems

1. In each of the following, x and y denote certain variable quantities. Discuss whether or not y is a function of x. If y is a function of x, can you write a formula that describes the relationship? Also, find the domain and range of each function.

   (a) $x =$ speed of a train; $y =$ distance the train travels in two hours

   (b) $x =$ height above sea level; $y =$ atmospheric pressure

   (c) $x =$ time of the year; $y =$ distance from the earth to the moon

   (d) $x =$ temperature at the Great Falls airport; $y =$ time of the day

   (e) $x =$ length of the side of a square; $y =$ area of the square

   (f) $x =$ area of a circle; $y =$ circumference of the circle

   (g) $x =$ weight of a letter; $y =$ first class postage for the letter

   (h) $x =$ OPEC price for a barrel of oil; $y =$ Dow Jones Industrial Average

2. A projectile was shot vertically into the air. The height h of the projectile was measured at 20 different times t. The following table gives the results, where t is in seconds and h is in meters.

Time (t)	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Height (h)	0.00	11.6	22.1	31.2	39.2	46.0	51.5	55.7	58.8	60.6	61.3
Time (t)	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	
Height (h)	60.6	58.8	55.7	51.5	46.0	39.2	31.2	22.1	11.6	0.00	

   (a) Graph this data.

   (b) Graph this data with lines connecting successive points. Do you think this is a reasonable approximation to the graph of h as a function of t?

3. Identify the domain of each of the following functions.

   (a) $f(x) = x^2 - 6x$

   (b) $g(x) = \sqrt{x^2 - 9}$

   (c) $f(t) = \sqrt{t^2 + t - 6}$

   (d) $h(x) = \dfrac{3}{x^2 + 6x + 8}$

   (e) $f(s) = \dfrac{41}{s^2 - 9}$

   (f) $y(t) = \dfrac{3}{\sqrt{3 - t^2}}$

   (g) $z(s) = \dfrac{1}{\sqrt{s^2 + 6s + 1}}$

   (h) $f(x) = \dfrac{4}{x(x^2 + 1)}$

4. For each of the following pairs of functions, find $f \circ g(x)$, $g \circ f(x)$, $f \circ g(3)$, and $f \circ g(3)$, when they are defined.

   (a) $f(x) = 4x + 12$, $g(x) = 5x - 2$

   (b) $f(x) = x^2 - 12$, $g(x) = \sqrt{x}$

   (c) $f(x) = 6x - x^2$, $g(x) = \dfrac{1}{x - 9}$

5. (a) If the graph of f is a straight line with slope 3 and the graph of g is a straight line with slope 4, show that the graph of $f \circ g$ is a straight line with slope 12.

   (b) If the graph of f is a straight line with slope m and the graph of g is a straight line with slope n, show that the graph of $f \circ g$ is a straight line with slope mn.

6. Graph each of the following functions on the given interval.

   (a) $f(x) = x^2 + 6x + 1$ on $[-10, 5]$

   (b) $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ on $[-5, 5]$

   (c) $f(x) = x^5 + 8x^4 + x^3 + x^2 + x + 1$ on $[-5, 5]$

   (d) $f(x) = x^5 + 8x^4 + x^3 + x^2 + x + 1$ on $[-10, 10]$

   (e) $g(t) = \dfrac{1}{1 + t^2}$ on $[-10, 10]$

   (f) $f(t) = \dfrac{t}{1 + t^2}$ on $[-10, 10]$

(g) $g(t) = \dfrac{t^2}{1+t^2}$ on $[-10, 10]$

(h) $g(x) = \dfrac{x^2-1}{x^4+1}$ on $[-5, 5]$

7. Another problem in graphing using a computer or a calculator arises when the function in question is not defined at some of the points in the interval of interest. For example, try graphing
$$f(x) = \frac{1}{x}$$
on the interval $[-5, 5]$. There are several problems which may arise. First, if the computer program tries to evaluate f at $x = 0$, you may get an error message. In this case you may have to change the number of points being plotted so that $x = 0$ is missed. Second, if the program evaluates f for values of x very close to 0, the output from the function will be very large. The result might be that the vertical scale of your graphing window is much too large. Hence, you may wish to change the scale of the vertical axis. Another problem that may occur arises because the graph of f has two pieces; that is, the part of the graph to the left of the y-axis is not connected to the part of the graph to the right of the y-axis. If your graphing program simply connects points as it moves from left to right, it will connect points on opposite sides of the y-axis which should not be connected. This may be hard to avoid with some software, but you should be aware of the problem and, consequently, interpret your results with care.

8. In light of the remarks in Problem 7, try graphing the following functions.

(a) $h(s) = \dfrac{1}{1-s^2}$ on $[-4, 4]$

(b) $g(x) = \dfrac{x^3}{1-x^2}$ on $[-4, 4]$

9. Recall that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Let $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

(a) What is the domain of f? What is the range of f?

(b) What is the domain of g? What is the range of g?

(c) Graph f and g on the interval $[-5, 5]$.

(d) Graph $h(x) = \lfloor x^2 \rfloor$ on the interval $[-2, 2]$.

10. We say a function f is *periodic* if there exists a constant T such that $f(t+T) = f(t)$ for every value of t in the domain of f.

(a) Is it possible for a polynomial to be periodic?

(b) Are any of the functions in Problem 1 periodic?

(c) Suppose x represents the number of days since January 1, 1950, and y represents the amount of rainfall in Spokane on day x. Do you think y is a periodic function of x? If not, might it in some way be close to periodic?