THE HEATH ELECTRONIC ANALOG COMPUTER

ITS USAGE FOR THE SOLUTION OF

ENGINEERING PROBLEMS

A THESIS

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ABSTRACT

The thesis is a tutorial paper on the basic principles of analog computers and analog computation. A systematic method for amplitude and time scaling is described. A number of sample problems which were worked on the Heath Electronic Analog Computer are described and results are shown. Through the examples, several interesting aspects of the theory of simulating differential equations on analog computers are discussed. Brief mention is made of special techniques and circuits which are used with analog computers.

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PREFACE

This paper is a report of an investigation made into the principles and uses of electronic analog computers, or as they are otherwise known, electronic differential analyzers. More specifically, an investigation was made of the use of the Heath Electronic Analog Computer.

It must be kept in mind throughout the reading of this paper that the investigation was made only into the principles of electronic analog computers. As such, no attempt was made to analyze the accuracy obtainable from the Heath computer, or to dwell excessively on the circuitry of the computer, except in instances where it was necessary to demonstrate some principle.

The author wishes to extend thanks to all those who have been of assistance, especially to his wife, Patricia, for her patient vigil throughout the long months that this work has been in preparation.

CHAPTER I

THE PRINCIPLES OF ANALOG COMPUTERS

A. TYPES OF COMPUTERS

In a broad sense, computers may be divided into two classes; digital and analog. The former, typified by the desk calculator and many of the large scale electronic computers, generally solves problems by numerical methods. Multiplication is performed by repeated additions, and integration is performed by summation. In analog computers, electronic and mechanical components are used to make physical quantities obey mathematical relations analogous to those in the original problems. For example, in the slide rule, the physical quantity of length simulates a mathematical variable; in the Heath computer, dc voltages represent the variables of a physical system.

In general, digital computers are more accurate and more expensive than analog computers. They solve problems in discrete steps of the input, intermediate, and output quantities, whereas in the analog computer the variables may be considered continuously varying quantities. On the other hand it may be philosophically argued that the variables in an analog computer are also discrete, their granularity being determined by such things as the difference in resistance between two adjacent turns on the winding of a wire wound potentiometer, and the least count to which meters may be read.

B. ANALOG COMPUTERS

Analog computers consist of various types. One type of machine, the network analyzer, produces direct simulation of electrical systems with L, C, and R components. The high cost of such a machine, however, has made it rather obsolete. Another type is the mechanical differential analyzer, more commonly known as the ball and disc integrator; and still another is the electronic differential analyzer, or electronic analog computer. In addition to the general characteristics of analog computers, this latter type is characterized by the use of the operational amplifier. The Heath Electronic Analog Computer is an example of this type.

One of the principal uses of an analog computer is to study the behavior of a real physical system by means of simulation. It may be quite impossible or impractical to study the real system itself. For example, it may be desirable to study the behavior under various operating conditions of a new aircraft design prior to construction of a model of the aircraft.

Simulation on an analog computer of a real physical system may be carried out by either of at least two basic philosophies (or a combination of these). For the first case, suppose that a system exists which consists of a number of elements, it being possible to mathematically describe the behavior of each of these elements individually. Then each of these elements can be simulated with analog components, and these components interconnected to simulate the whole system. The response of the system to various input, or driving, conditions may then be studied; or the simulation model may be adjusted to obtain a desired response for a particular input function. This will indicate the characteristics that the real system must possess to perform as desired.

The second basic method consists of first describing the operation of the whole system by a set of differential equations, and then solving these equations on the computer. While this method usually performs the simulation with a smaller number of analog components than if each element of the real system is simulated individually, it has the disadvantage that in general, specific components of the computer are not associated with specific elements of the real system. Rather, the adjustment of an analog component changes a coefficient in one of the differential equations being simulated. This coefficient may reflect the value of several elements of the system under study.

The utility of analog computers and techniques is not restricted to the simulation of physical systems. Special purpose computers, which may often be conveniently assembled from standard components of commercially available machines, can themselves serve as control system elements in some applications. For example, a single commercial multipurpose computer might be adapted to process signals controlling several phases of an industrial process. It has been suggested[1] in

[1]Korn and Korn, page 110. (Source 1)

one case, that analog computers be used for continuous recomputation of optimum set points as functions of the composition or quality of raw materials entering a process. It is conceivable that such techniques would permit the use of less pure or cheaper raw materials.

C. COMPONENTS AND OPERATIONS OF ELECTRONIC ANALOG COMPUTERS

The heart of an electronic analog computer is the operational amplifier. The operations most commonly performed on the operational amplifier are summation and integration. Specialized circuits can be used to accomplish a variety of other functions.

The operational amplifier is a very high gain (30,000 to 50,000 in the Heath) device. Figure 1a shows the symbol for an adder. The figures inside the adder are the gains associated with the various inputs. Figure 1b shows the use of the operational amplifier and the associated components needed to implement an adder. The gain equations are also shown. Each amplifier in an analog computer is numbered. It is customary, when drawing an analog computer setup, to show the number of each amplifier used, as in Figure 1b. Figures 2a and 2b show the equivalent notation for an integrator.

Another component used in the electronic analog computer is the potentiometer. Potentiometers are used as adjustable voltage dividers to multiply a variable by a positive constant less than one. The use of potentiometers will be evident with the examples given later.

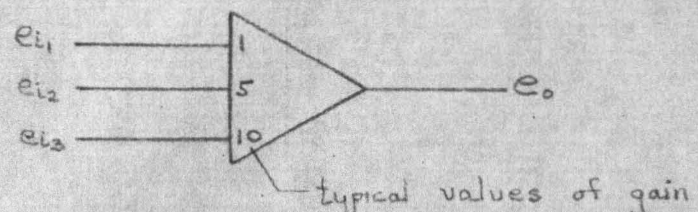
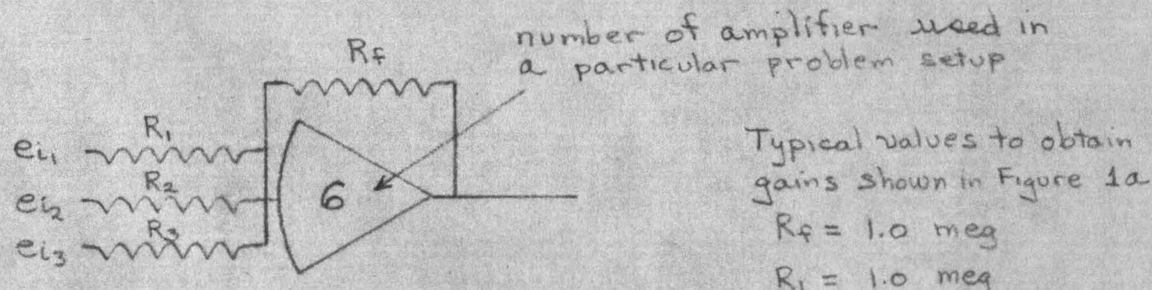


Figure 1a. Symbol for Operational Amplifier and Associated Circuitry When Used as an Adder.



$$e_o = -R_f\left[\frac{e_{i_1}}{R_1} + \frac{e_{i_2}}{R_2} + \frac{e_{i_3}}{R_3}\right]$$

Figure 1b. Operational Amplifier and Associated Components for Implementation of Figure 1a.

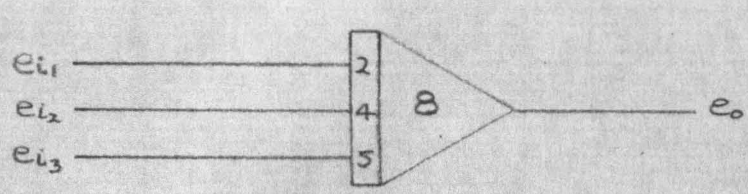
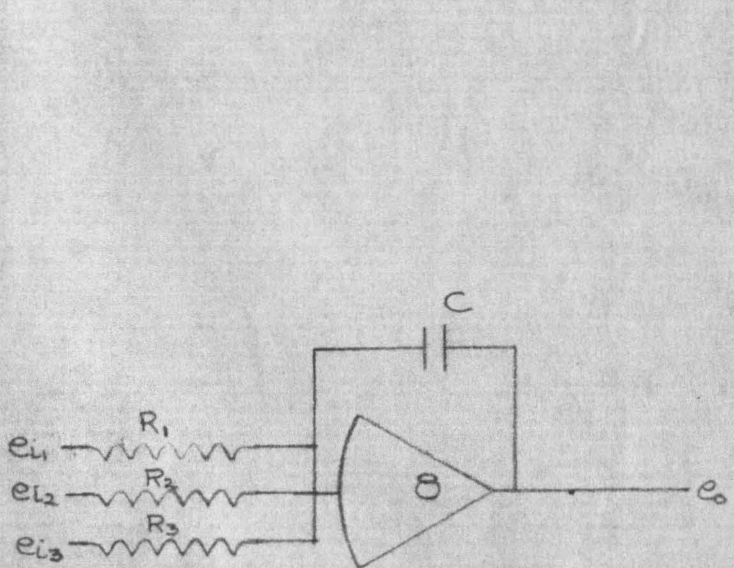


Figure 2a. Symbol for Operational Amplifier and Associated Circuitry When Used as an Integrator.



Typical values to obtain gains shown in Figure 2a

$C = 1.0\ \mu F$

$R_1 = 0.5$ meg

$R_2 = 0.25$ meg

$R_3 = 0.2$ meg

$$e_o = \frac{-1}{C}\int_0^t \left(\frac{e_{i_1}}{R_1} + \frac{e_{i_2}}{R_2} + \frac{e_{i_3}}{R_3} \right) dt$$

Figure 2b. Operational Amplifier and Associated Components for Implementation of Figure 2a.

A servomultiplier is a device which produces an output proportional to the product of two input variables. Several linear potentiometers are grouped together and driven by a servomotor. A reference voltage is applied to one of the potentiometers and the output from the arm of this potentiometer is subtracted from the first input variable. This "error" voltage is sent through a high gain servoamplifier and applied to the servomotor which mechanically drives the potentiometer in the direction to reduce the "error" to zero. This is in effect a servomechanism. The second input variable is applied to a second potentiometer, and a voltage proportional to the product of the two variables is available at the arm of this potentiometer. Additional variables may be applied to additional potentiometers and in each case the output voltage is proportional to the product of the common first variable and the individual second variables.

The servo resolver is a special kind of servomultiplier. In the servo resolver a precision sine-cosine potentiometer, which delivers output voltages proportional to the sine or cosine of the brush angle, is used instead of a multiplying potentiometer as in the servomultiplier. Two such potentiometers are mounted concentrically within each resolver. The name resolver is derived from the action of the device as a converter of a vector displacement into its rectangular components.

It is to be noted here that the Heath Electronic Analog Computer contains neither servomultipliers nor servo resolvers. The comparatively high speeds with which this computer solves problems when in the repetitive mode (section ID) would not permit their use.

Another component widely used in electronic analog computers is the vacuum diode. This has a variety of uses including limiting, and the generation of many special functions.

Initial condition power supplies are used for setting initial conditions on integrators.

Relays in analog computers find application in control circuits. Such circuits are used to start or hold a problem. The use of relays will be made clearer by the examples of this paper.

D. REPETITIVE AND NONREPETITIVE OPERATION

Although many analog computers are designed to solve a problem only once when integration is permitted to start, some computers are designed to solve problems on a repetitive basis, repeating the solution at a rate of anywhere from 0.1 to 50 solutions per second. The Heath Electronic Analog Computer can be used in either mode of operation. In the repetitive mode it repeats solutions at an adjustable rate between 0.6 and 6 cps.

The general characteristics of repetitive computers may be enumerated as follows:

1. Problems are usually solved in a more compressed time scale than on nonrepetitive computers. In general this means that the operational amplifiers are run at higher gain.
2. It is practical to display the solutions on an oscilloscope. This makes it possible to note immediately the effect on the solution of varying the system parameters.

3. The choice of relatively slow speed computing elements is restricted. This means that servomultipliers and servo resolvers are not used. Hence, these devices for multiplication and trigonometric function generation are not found in the Heath Computer. Special techniques must be used if multiplication of two variables is desired.

4. They are usually only moderately accurate and are somewhat inexpensive. This is due in general to the relatively low open loop gain requirements of the operational amplifiers, and the less expensive capacitors which may be used. Usually mica or ceramic capacitors are adequate for integration. Integrating capacitors in high accuracy (non repetitive type) computers should be polystryene or of some similar plastic in order to achieve the high leakage resistance necessary.

CHAPTER II

THE HEATH ELECTRONIC ANALOG COMPUTER

A. COMPONENTS

The Heath Electronic Analog is equipped with fifteen operational amplifiers. Two potentiometers are associated with each amplifier, and a bridge circuit is provided to set the potentiometers allowing for the effect of amplifier loading on the potentiometer. Two additional potentiometers not associated with the amplifiers each have a vernier dial which allows them to be set at any desired value. The resistance of all potentiometers is 100,000 ohms.

Six independent initial conditions power supplies in the computer may be set at any voltage from -100 to + 100. A reference supply delivers plus or minus 100 volts.

Four relays, each with four transfer contacts are provided. The relays may be controlled manually from switches on the front panel or may be driven by a repetitive oscillator at a rate adjustable from 0.6 to 6 repetitions per second. In addition, the relay windings are brought out to jacks on the front panel for other wiring options.

Finally, eight vacuum diodes are included in the Heath Computer.

B. OPERATIONAL METHODS

Before using an analog computer it is necessary that the operational amplifiers be adjusted to give zero output with zero input. The method of accomplishing this is illustrated in Figure 3. The amplifier is simply switched into the circuitry shown and the null control on the amplifier is adjusted to give zero output with a grounded input. The specific method of accomplishing this by the various switches on the front panel of the computer is adequately covered in the Heath instruction manual.

When setting potentiometers for the purpose of delivering a particular fraction of a voltage to the input of a summing or integrating circuit, it is important to account for the loading effect of the input impedence of the summing circuit. Fortunately, it is not necessary to hand compute this effect; a built-in bridge circuit makes it possible to set the potentiometers taking the loading effect into account. The principle of accomplishing this is shown in Figure 4. R_f and R_i are externally applied resistors. It is to be emphasized here that the value of R_i used for setting the potentiometer must be the same as the value of the corresponding resistor to be used in the setup for the problem situation. Suppose that the problem called for a potentiometer setting at 0.432 feeding an amplifier with a gain of 10. Typically, for this case, R_f is 1.0 megohm and R_i is 0.1 megohm.

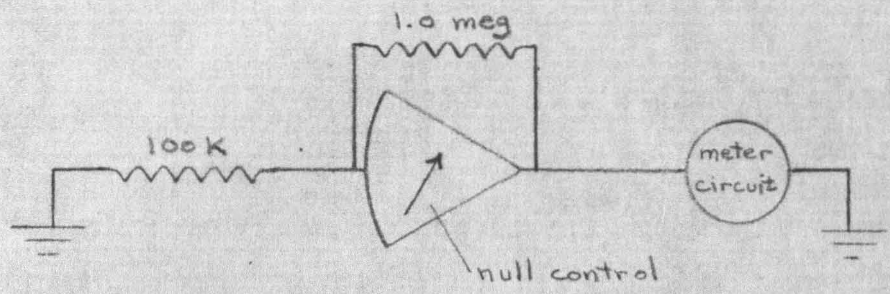


Figure 3. Circuit for Adjusting Null Control on Operational Amplifier.

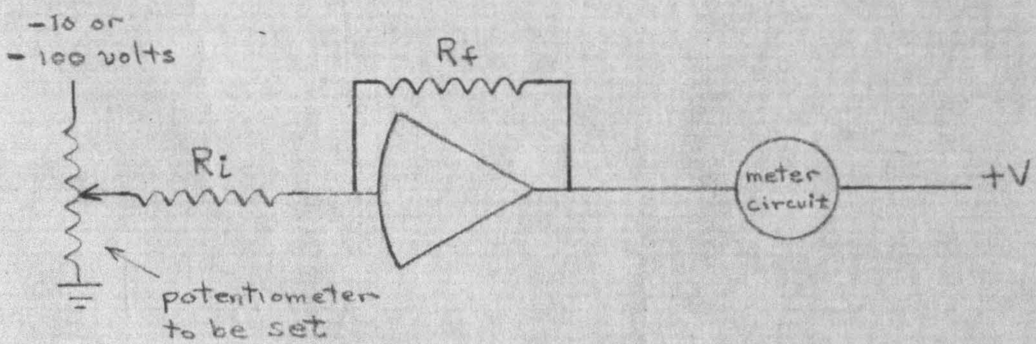


Figure 4. Method of Setting Potentiometer Taking Loading Effect Into Account.

These values of resistor are plugged into the computer, -10 volts is applied to the potentiometer, and + V is adjusted to 43.2 volts. The potentiometer is then adjusted until the meter reads zero. The details of setting potentiometers are covered in the instruction manual.

C. RELAY CIRCUITS

Relays are used in analog computers for control purposes such as setting initial conditions. When an analog computer is used in the repetitive mode, the initial conditions are reset for each cycle of operation. This is accomplished by driving the control relays from a relay on the repetitive oscillator. The interconnection of the components is best described by Figure 5. Their operation is evident from the drawing.

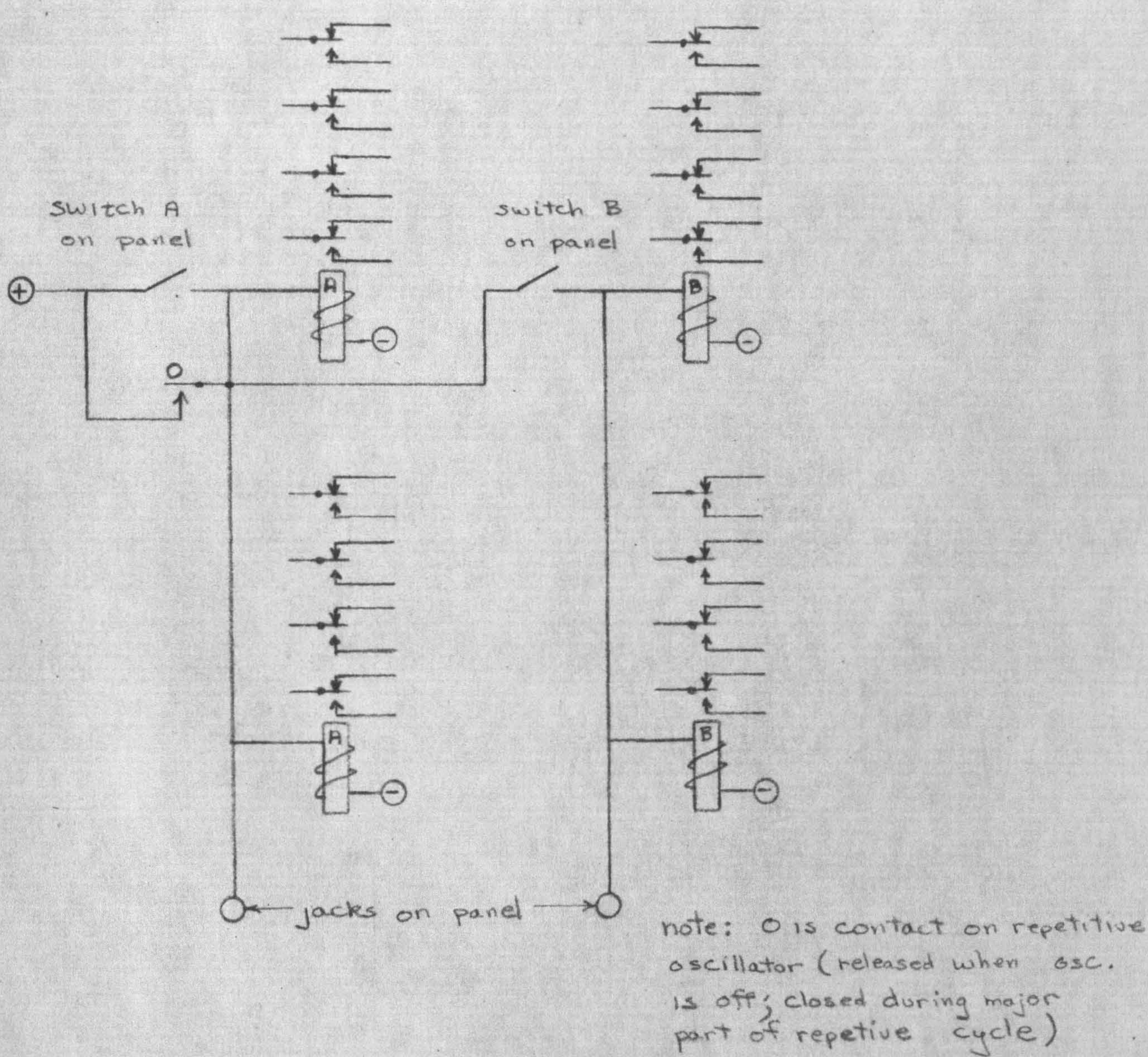


Figure 5. Equivalent Circuit for Operation of Computer Relays.

CHAPTER III

THE PRINCIPLES OF ANALOG COMPUTATION

A. BASIC PRINCIPLES OF PROBLEM SOLUTION

The usual first step in solving a differential equation on an analog computer is to write the highest derivative of the dependent variable of the equation in terms of the lower derivatives and whatever constants or functions of the independent variable may be included in the equation. For the present, time is considered as the only independent variable encountered in the equations; the extension of the method to other independent variables is discussed later.

The next step is to assume that the highest order derivative is available at the output of an operational amplifier. Each integrating circuit following this point produces (with sign changes) successively lower derivatives. These derivatives, in turn, are fed back with correct sign and amplitude, to the input of the first summing amplifier. Circuits are available for differentiation, but it is a noisy process and should be avoided whenever possible.

It now remains to connect the external inputs to the problem, and to set the proper initial conditions at the output of each integrator. In general, this means wiring an isolated voltage, representing the initial value of a variable, in series with a switch (usually a relay contact) across each integrator. The switch opens at the start of the problem. To avoid noise troubles, the relay should be at the input end of the amplifier, and the voltage at the output end. In this way, any voltages due to leakage currents in the initial conditions power supplies will be applied (after the relay contacts have opened) to the output of the amplifier rather than to the input where their effect would be far greater. If the initial condition of a variable is zero, the relay contact is wired directly across the integrating capacitor.

B. AMPLITUDE AND TIME SCALING

Although at first glance, it might seem extremely convenient to set up all analog computer problems so that one volt corresponds to one unit in the variable being investigated, and one second of machine time corresponds to one second of problem time, a second glance will show that such niceties are sometimes neither practical nor realizable. If a particular variable is expected to range in value, for instance, from 0 to 200 feet, and one were to say that one foot equals one volt on a computer where the maximum output of an amplifier is 100 volts, there is an obvious inconsistency. A real time solution of a differential equation describing the population growth of a city would take years of computer time.[1] In many instances it is desirable to connect the output of an analog computer to a pen recorder. If there are frequencies greater than two or three cycles per second to be recorded, the pen will not follow with the desired accuracy.

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[1]In an article by Smith and Erdley (Source 6) an economic system is simulated. One year of real time is made analogous to 200 microseconds of machine time.

The above examples are meant to illustrate the need for both amplitude and time scaling of problems so that they fit the analog computer properly and reasonably. A systematic method of doing this is desirable.

A number of methods are in use for amplitude and time scaling. The method described herein has been worked out by the author, but no claim is made to originality.

The method introduces the concept of a machine unit. One machine amplitude unit is the maximum allowable voltage at the output of an operational amplifier. Hence, the output of an operational amplifier must not be allowed to exceed one machine unit. For many analog computers (including the Heath) this is 100 volts. One machine time unit is one second.

Using this method for scaling, the general procedure is to transform, by a set of suitable transformation equations, each of the variables of the original equation (or equations) into machine variables. The result is a machine equation which can be implemented on the computer directly. The amplitudes and times of solution will be correct. When the machine solution is obtained, the inverse of the original transformation equations are used to obtain the answer in terms of the original problem variables.

The notation adopted for this method uses lower case letters for the problem variables and upper case letters for the machine variables. Time derivatives of the problem variables are represented by the operator notation p, p^2, etc., in lower case letters. The same notation, but with upper case letters is used for the machine time derivatives of the variables.

The procedure for writing the transformation equations and machine equations is as follows:

1. Write the equation or equations to be solved with the proper coefficients. If a coefficient is to be varied note the range of the variation.
2. Estimate the maximum expected value of each of the variables. Methods of estimating these values are available in the literature.
3. Decide tentatively on the relative time of solution to be used in the problem. This can be readjusted later if it appears that a different relative time will make the simulation easier.
4. Considering now amplitude scaling of each of the dependent variables, determine a scale factor such that:

   $$a_i \leq \frac{1}{|\text{maximum expected value of variable, i}|}$$

   For example, if the maximum abxolute value of py is not expected to exceed 4 feet/second, then:

   $$a_{py} = 0.25 \text{ machine units/feet/sec.}$$

   It is important to point out here that the value of a_{py} is not necessarily related to that of a_y, $a_{p^2 y}$, etc.
5. For the time variable write $p = \alpha_t P$, or $T = \alpha_t t$. In this equation α_t represents the factor by which a problem is slowed down or the time of solution increased. By this definition, a value of α_t less than 1 indicates that problem solution time is decreased.

6. Using the scale factors of steps 4 and 5, the transformation equations are written using the relationship:

$$\text{problem variable} = \frac{(\alpha_t)^k}{a_i} \text{ machine variable,}$$

where k is the order of the derivative of the dependent variable. For example, if $a_{py} = 0.25$ machine units/foot/sec. and $\alpha_t = 10$, then

$$py = \frac{10^1 PY}{0.25}, \quad py = 40\ PY$$

If the problem is solved in real time, then the transformation relationship reduces to:

$$\text{problem variable} = \frac{\text{machine variable}}{a_i}$$

It should be clearly understood that these transformations are performed on the variables; not their coefficients.

7. Replace each of the variables in the problem equation by their equivalent in machine language. The machine equation has now been formulated.

8. Write the relationship between the various derivatives of each variable if this relationship is other than one to one.

After the above is carried out, the problem is ready to be put on the computer. These rules will be clarified in the next section by the examples that were actually worked on the computer.

C. MISCELLANEOUS CONSIDERATIONS

As a final word on scaling, two miscellaneous items require clarification.

Suppose that the independent variable of a differential equation is not time, but is, for example, x. The most straightforward way of handling this situation is to pretend that the variable is t. The standard rules apply, and at the completion of the solution it is merely necessary to substitute x for t.

The remaining situation is the case where the independent variable occurs explicitly in the differential equation. For example:

$$py = -y + t - \sin 2t$$

In this case it is necessary to define a new machine variable τ defined by the transformation equation

$$\tau = a_t t$$

The scale factor a_t must not be confused with the time scale factor α_t. τ is generated as a function of T which is the independent variable in the machine. It is therefore proper to consider τ as a dependent machine variable.

CHAPTER IV

EXAMPLES OF ANALOG COMPUTER PROBLEMS

This chapter illustrates the use of the analog computer by solving some typical problems on the computer.

A. LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

The first example consists of the solution of several somewhat similar linear second order differential equations. They are inserted here to illustrate in particular the proper use of amplitude and time scale factors. The equations to be solved are:

$p^2y + 2y = 200$ (in real time)

$p^2y + y/5 = 20$ (in real time)

$p^2y + 2y = 200$ (in less than real time)

A1. Consider first the equation

$$p^2y + 2y = 200 \qquad (1)$$

This may be written

$$py = \int_0^t (200 - 2y)dt \qquad (2)$$

It is estimated that the maximum values of y and py do not exceed 200. One need not be concerned about the maximum value of p^2y, since if equation (2) is solved, p^2y never appears. Hence, the following scale factors may be formulated.

$$a_y = \frac{1}{200}$$

$$a_{py} = \frac{1}{200}$$

A logical choice appears to be to run the problem in real time.

Hence:

$$t = T \text{ or } p = P$$

The transformation equations are consequently:

$$y = 200\ Y$$

$$py = 200\ PY$$

Equation (2) may now be written

$$200\ PY = \int_0^T (200-400Y)\ dT$$

or $$PY = \int_0^T (1-2Y)\ dT \qquad (3)$$

Figure 6a shows a block diagram of the logic to solve equation (3). Figure 6b shows the implementation and the amplifiers used to run this problem. The quantities Y and -PY were displayed on the scope. The results obtained (with -PY inverted) are shown in Figure 6c.

Consider for the moment equation (3). This may be solved by classical methods.

$$PY = \int_0^T (1-2Y)dT \qquad (3)$$

$$p^2Y + 2Y = 1 \qquad (4)$$

$$m^2 + 2 = 0$$

$$m = \pm i\sqrt{2}$$

$$Y = A \cos\sqrt{2}\ T + B \operatorname{Sin}\sqrt{2}\ T + Y_p$$

Let $Y_p = C$

Substituting into (4)

$$2C = 1 \qquad C = 1/2$$

$$Y = A \cos\sqrt{2}\ T + B \sin\sqrt{2}\ T + 1/2$$

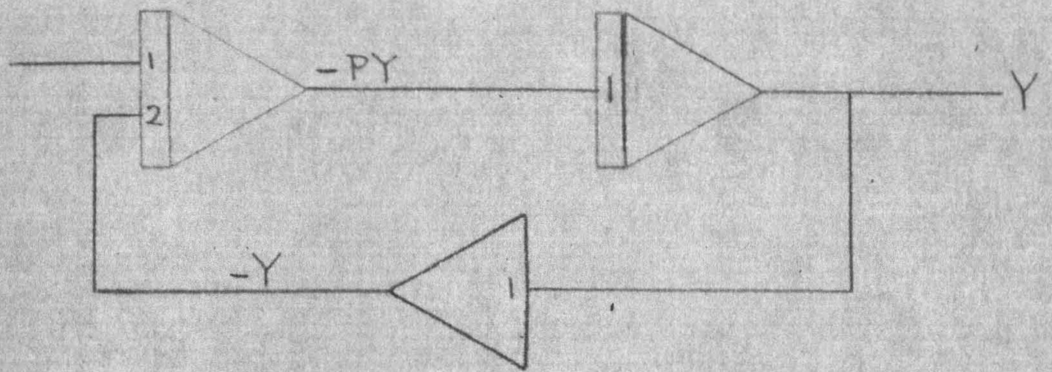

Figure 6a. Logic Diagram for Solution of Equation.

$$PY = \int_0^T (1-2Y)\, dT$$

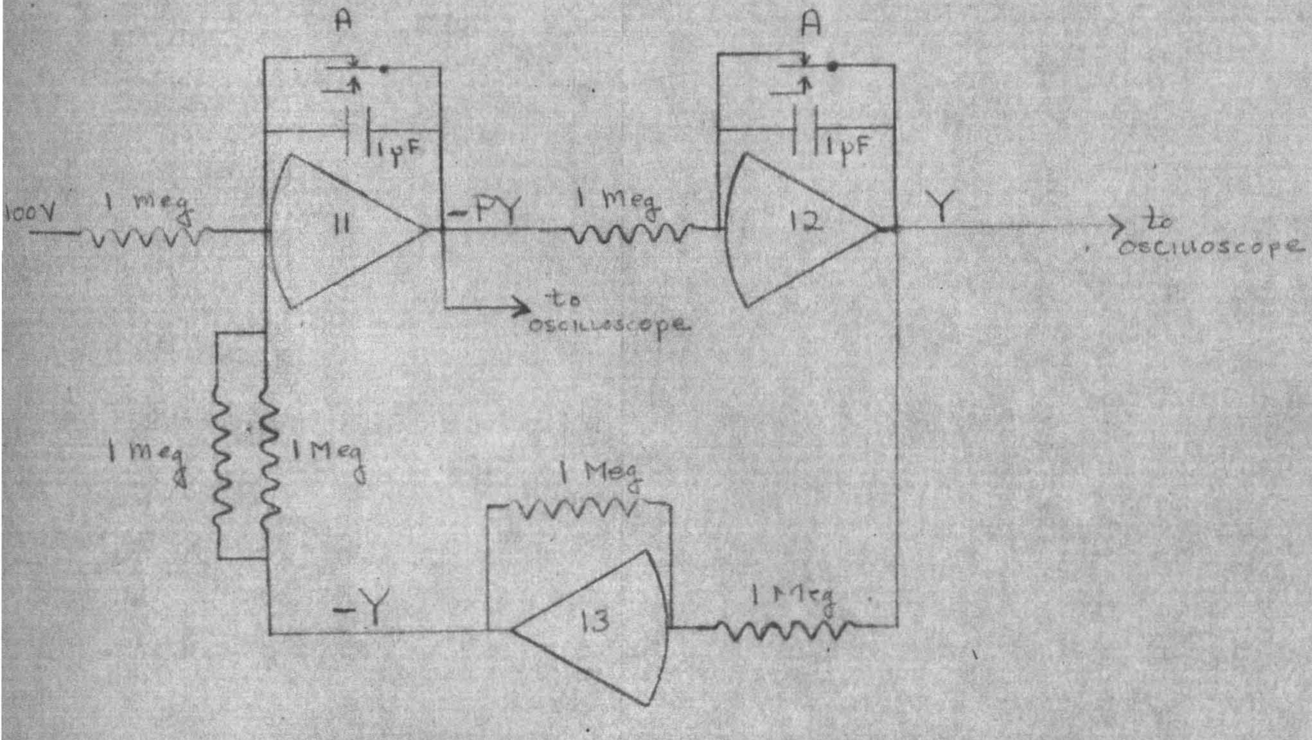

Figure 6b. Implementation of Figure 6a.

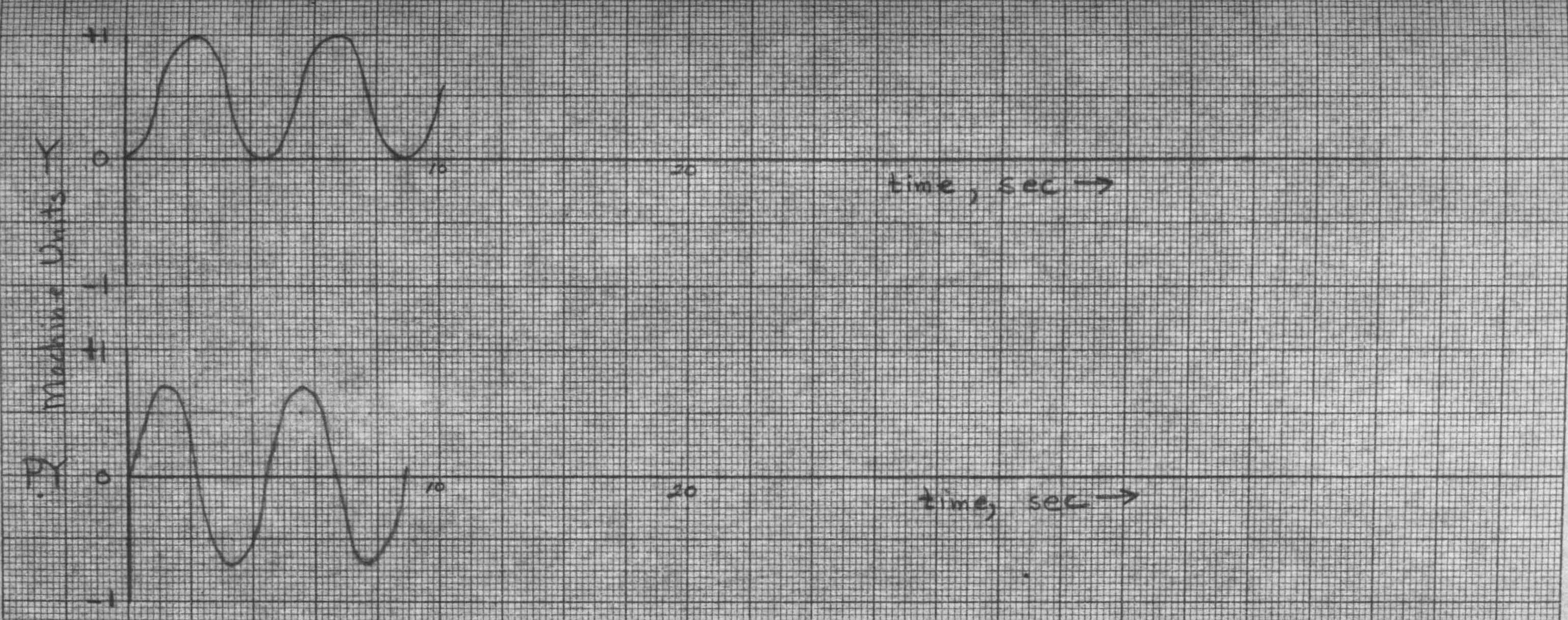

Figure 6c. Oscilloscope Observations of Operation of Figure 6b.

at $T = 0$ $Y = 0$ $A = -1/2$

$$PY = \frac{\sqrt{2}}{2} \sin\sqrt{2}\ T + B\sqrt{2} \cos\sqrt{2}\ T$$

at $T = 0$, $PY = 0$ $B = 0$

Hence: $Y = 1/2 - 1/2 \cos\sqrt{2}\ T$ (5)

and $PY = \frac{\sqrt{2}}{2} \sin\sqrt{2}\ T$ (6)

An inspection of Figure 6c shows that the curves fit the above solutions.

If the inverse of the transformations equations are now substituted into the solutions (5) and (6), the solutions are obtained in terms of the original variables.

$$y = 100 - 100 \cos\sqrt{2}\ t$$

$$py = 100\sqrt{2} \sin\sqrt{2}\ t$$

It is easily verified that these are the exact same solutions obtained if equation (1) is solved directly.

A2. Consider now the second equation:

$$p^2y + y/5 = 20$$

$$\text{or} \quad py = \int_0^t (20 - y/5)\, dt$$

Using the same scale factors as in the previous problem, the machine equation is:

$$PY = \frac{1}{10} \int_0^T (1-2Y)\, dT$$

Figure 7a shows the logic to solve this equation, and Figure 7b shows the implementation. Figure 7c shows the solutions for Y and PY recorded on the oscilloscope. It appears that the expressions

$$Y = \frac{1}{2} - \frac{1}{2} \cos\sqrt{\frac{2}{10}}\ t \quad \text{and}$$

$$PY = \frac{1}{2}\sqrt{\frac{2}{10}} \sin\sqrt{\frac{2}{10}}\ t$$

fit the curves of Figure 7c. Transforming, the solutions to the original equation are

$$y = 100 - 100 \cos\sqrt{\frac{2}{10}}\ t$$

$$py = 100\sqrt{\frac{2}{10}} \sin\sqrt{\frac{2}{10}}\ t$$

<u>A3</u>. Consider again the equation of the first example:

$$P^2y + 2y = 200 \qquad (1)$$

This equation will now be solved in a manner such that one second of problem time is equal to 10 seconds of computer time, or $\alpha_t = 10$. Hence, the solution will take ten times as long. The scale factors to be used are as follows:

$$a_y = \frac{1}{200}$$

$$a_{py} = \frac{1}{200}$$

$$a_{p^2y} = \frac{1}{200}$$

$$p = 10\ P \qquad \text{or} \qquad T = 10t$$

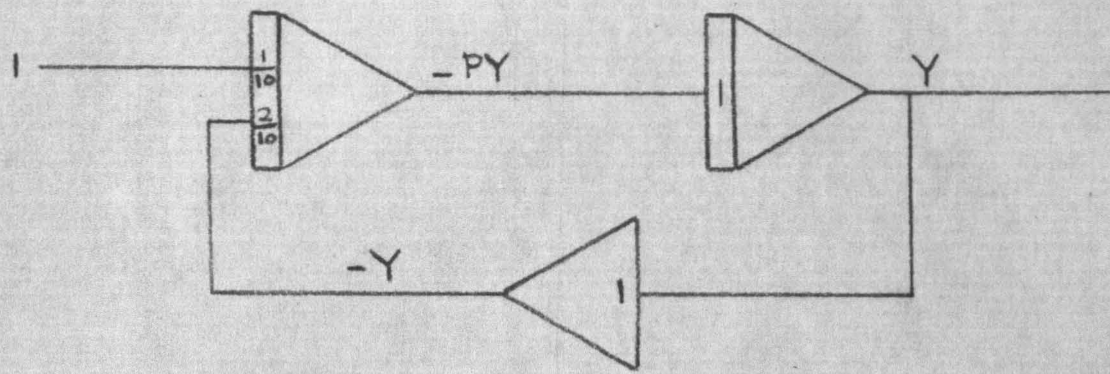

Figure 7a. Logic Diagram for Solution of Machine Equation

$$PY = \frac{1}{10} \int_0^T (1 - 2Y)\, dT$$

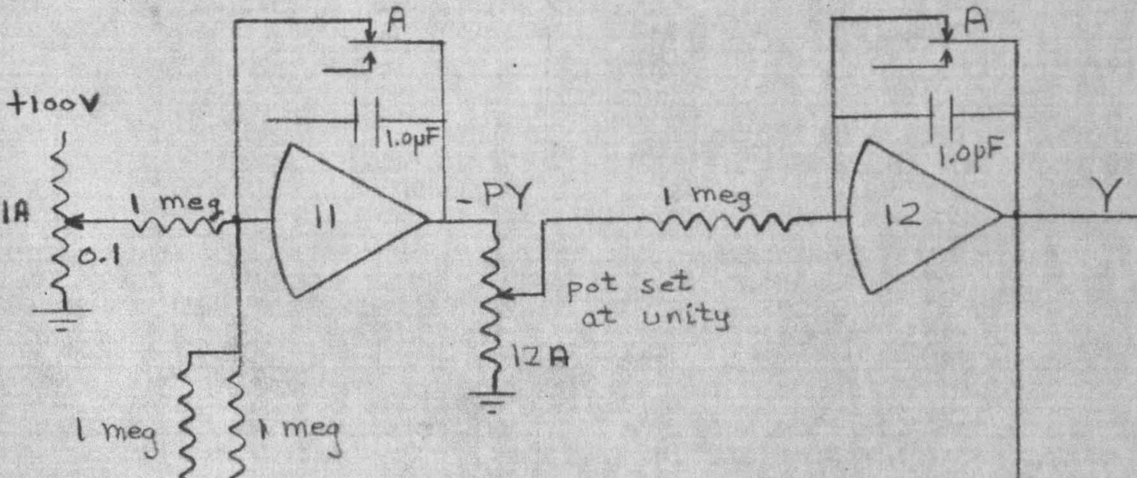

Figure 7b. Implementation of Figure 7a.

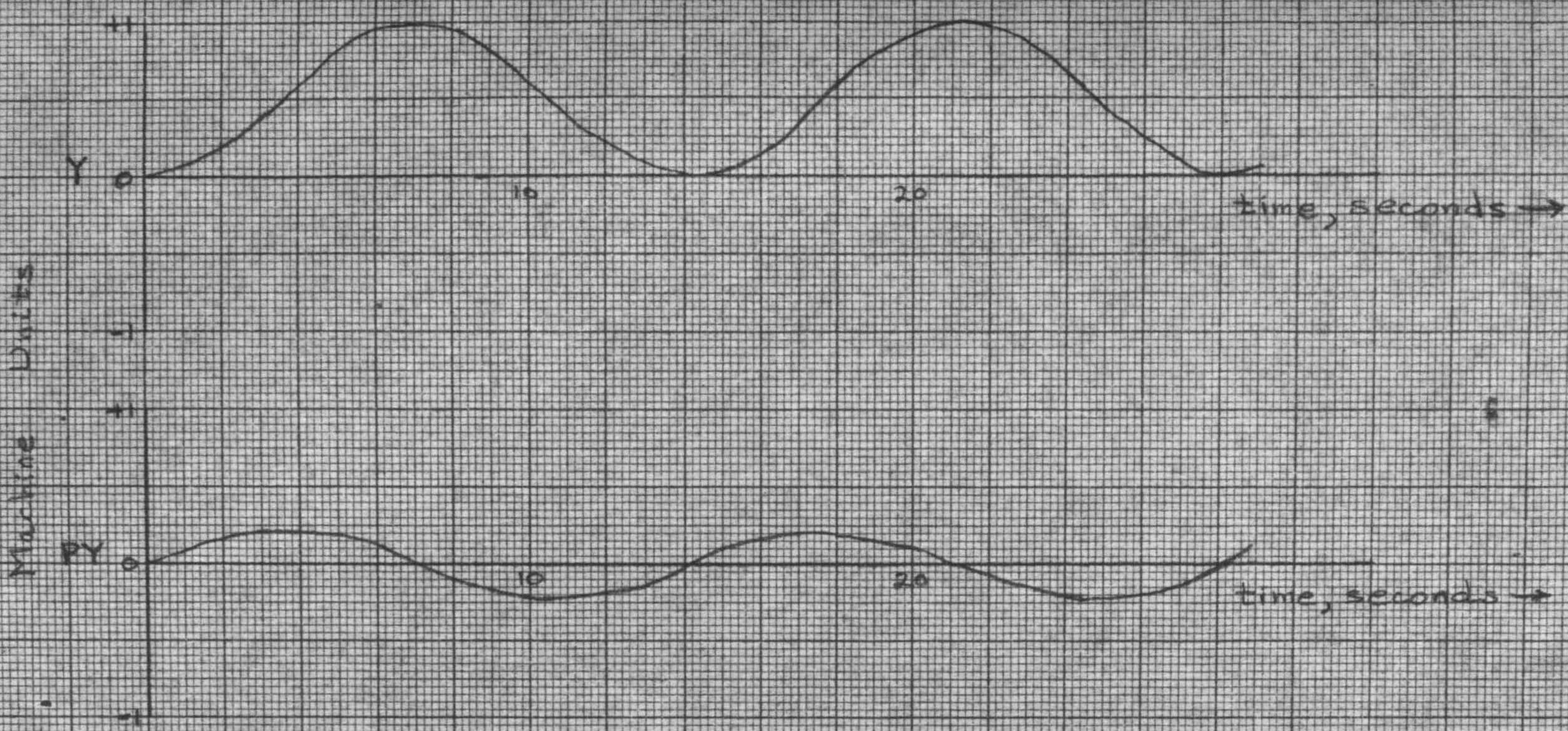

Figure 7c. Oscilloscope Observations of Operation of Figure 7b.

The transformation equations are:

$$y = 200\ Y$$

$$py = 2{,}000\ PY$$

$$p^2y = 20{,}000\ P^2Y$$

Substituting into equation (1),

$$PY = \frac{1}{100} \int_0^T (1-2Y)\ dT \qquad (7)$$

It would have also been possible to write

$$py = \int_0^t (200-2Y)dt$$

$$2{,}000\ PY = \int_0^T (200-400Y)\ \frac{dT}{10}$$

$$PY = \frac{1}{100} \int_0^T (1-2Y)\ dT \qquad (7)$$

Since we are really solving the same equation as in A1, the value of py ought to be the same, but the scale factors make the value of PY 1/10 of that obtained when the problem was run in real time. Accuracy considerations make it undesirable to run the output of operational amplifiers at such low voltages. Hence it is better to solve the equation

$$10PY = \frac{1}{10} \int_0^T (1-2Y)dT$$

Figures 8a and 8b show the logic and results obtained for this equation. The implementation is identical to that of Figure 7b with potentiometer 12a set at 0.1. The solution fits the equations

$$Y = \frac{1}{2} - \frac{1}{2}\ \cos\left(\frac{T}{\sqrt{50}}\right)$$

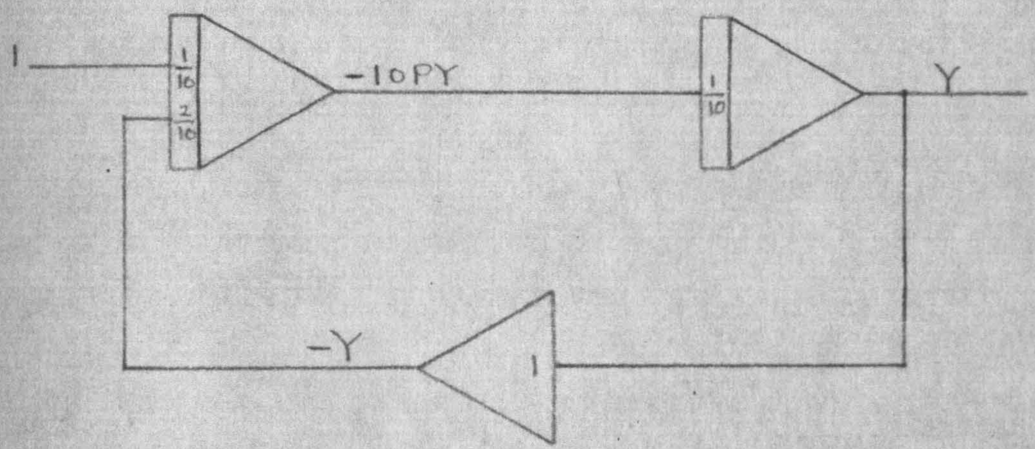

Figure 8a. Logic Diagram for Solution of Machine Equation.

$$10PY = \frac{1}{10} \int_0^T (1-2Y)\, dT$$

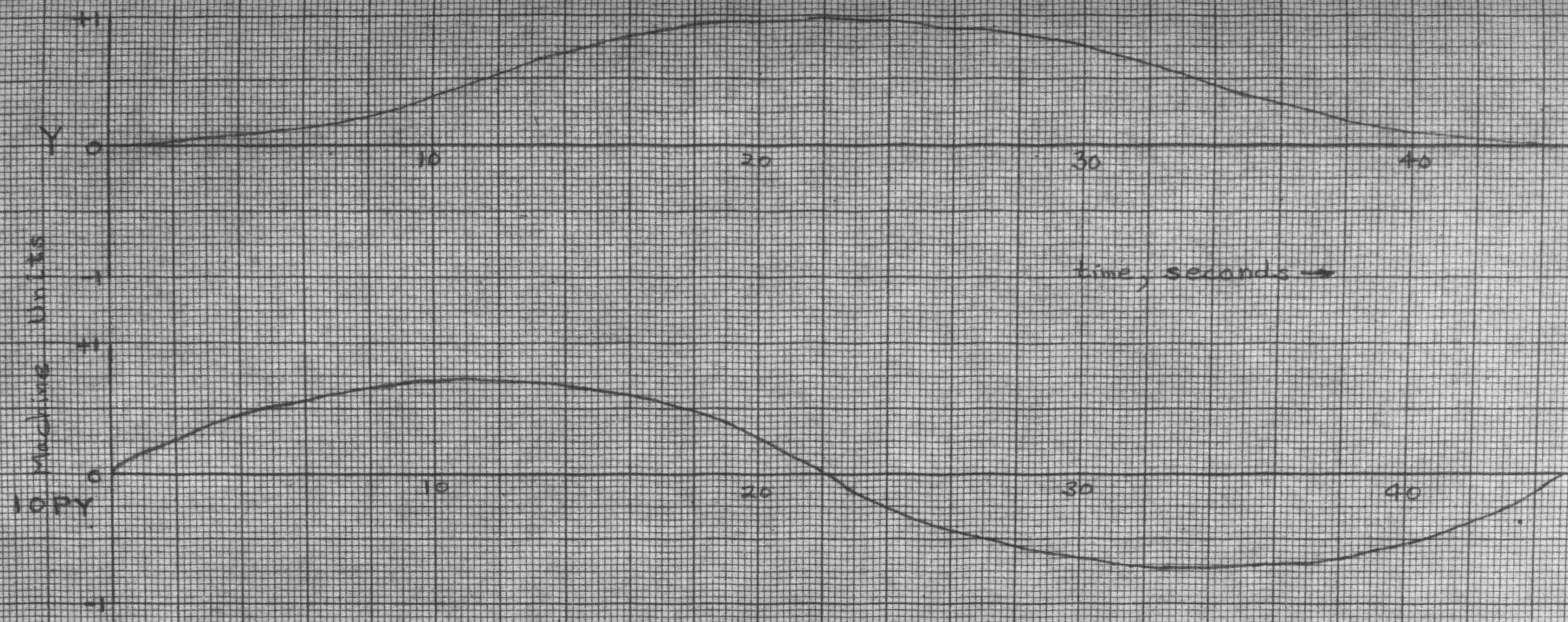

Figure 8b. Oscilloscope Observations of Logic of Figure 8a.

$$-10PY = -\frac{\sqrt{2}}{2} \quad \sin \left(\frac{T}{\sqrt{50}} \right)$$

$$\text{or} \quad PY = \frac{\sqrt{2}}{20} \quad \sin \left(\frac{T}{\sqrt{50}} \right)$$

In terms of the problem variables the solution is

$$y = 100 - 100 \cos \sqrt{2}t$$

$$py = 100 \sqrt{2} \quad \sin \sqrt{2} \quad t$$

It is satisfying to note that the same solution was obtained when the problem was solved in real time.

It is interesting to note that although PY does not have the same maximum value that it did when the problem was solved in real time, 10 PY in this solution has the same maximum value as PY in the real time solution, and in both cases Y has the same maximum value. This seems to suggest that if a problem is properly amplitude scaled when the time scale is changed, if PY is replaced by $\alpha_t PY$, P^2Y by $\alpha_t^{\,2}P^2Y$, etc., at the output of the integrators. This also suggests an alternative method of time scaling. The problem can be set up as planned for a real time solution, and the gain of all integrators then reduced by the factor α_t. The result shown in Figure 9 is the logic of Figure 8a with the parameter values of Figure 6a. In substituting into the equations for the solutions to this problem, however, the transformation equations

$$y = 200 \; Y$$

$$py = 200 \; PY \qquad \text{(rather than 2000 PY)}$$

$$T = 10 \; t$$

would be used.

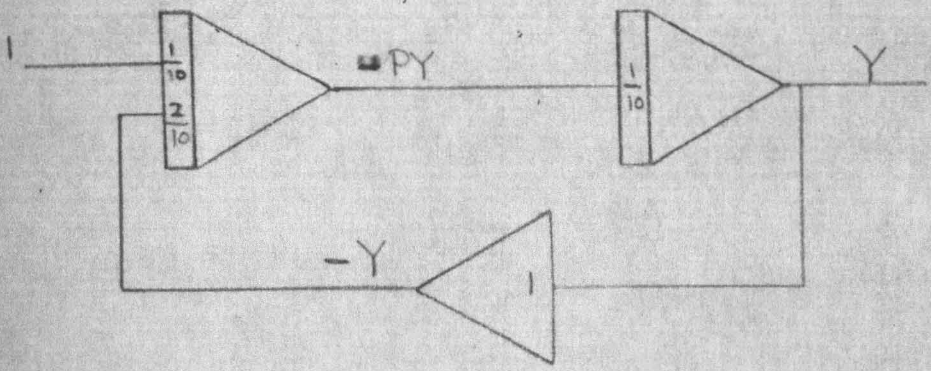

Figure 9. Figure 6a With Integrator Gains Reduced to $\frac{1}{10}$ Their Value.

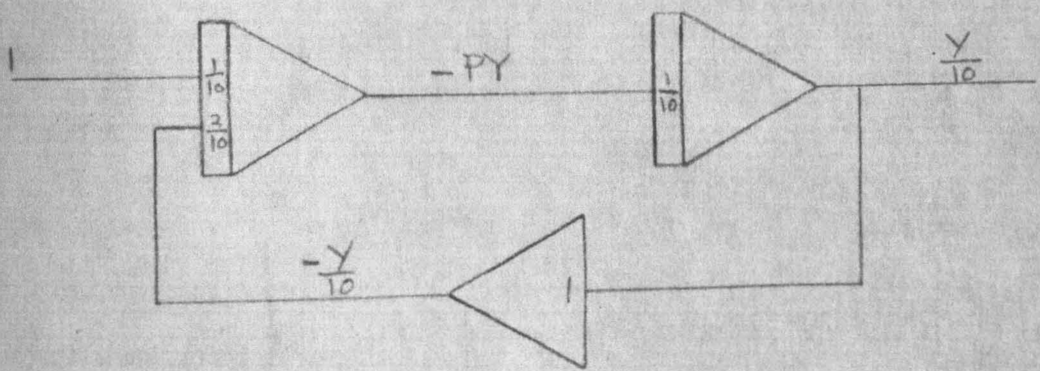

Figure 10. Logic for Solution of Machine Equation

$$PY = \frac{1}{10} \int_0^T \left(1 - \frac{2Y}{10}\right) dT$$

There is one other interesting aspect of this example which might be inserted here for academic interests. The logic of the last mentioned method of solution may be examined without reference to a specific problem or scaling. If the output of the first integrator is called - PY, and the output of the second called Y/10, as in Figure 10, it appears that the machine is solving the equation

$$PY = \frac{1}{10} \int_0^T \left(1 - \frac{2Y}{10}\right) dT$$

or $$P^2Y + \frac{2Y}{100} = \frac{1}{10}$$

The solution for this equation is:

$$y = 5 - 5 \cos \frac{T}{\sqrt{50}} \quad \text{or} \quad \frac{Y}{10} = \frac{1}{2} - \frac{1}{2} \cos \frac{T}{\sqrt{50}}$$

$$-PY = \frac{-\sqrt{2}}{2} \sin \frac{T}{\sqrt{50}}$$

These values for -PY and $\frac{Y}{10}$ are the outputs, respectively of the first and second integrators. They are identical to the values for the output of the first and second integrators of Figure 8a, although they were known by different names in that case. The fact that they are the same is not surprising, for examination shows the logic of Figure 8a to be identical with that of Figure 10.

B. LINEAR FIRST ORDER SIMULTANEOUS DIFFERENTIAL EQUATIONS

Consider the pair of simultaneous differential equations:

$$px + 4x + 4\,py + 10y = 6$$

$$x + py + 3y = 0$$

If, at $t = 0$, $x = 0$, and $y = 0$, the solutions are:

$$x = -12e^{-t} + 3e^{-2t} + 9$$

$$px = 12e^{-t} - 6e^{-2t}$$

$$y = 6e^{-t} - 3e^{-2t} - 3$$

$$py = -6e^{-t} + 6e^{-2t}$$

It is desired to show how these solutions may be obtained on the analog computer.

First it is estimated that the maximum absolute values of the variables involved do not exceed the following:

$$|x| \leq 10$$

$$|px| \leq 10$$

$$|y| \leq 4$$

$$|py| \leq 2$$

The transformation equations are:

$$x = 10X$$

$$px = 10PX$$

$$y = 4Y$$

$$py = 2PY$$

$$t = T$$

and since $y = \int py\,dt$; $\qquad Y = \frac{1}{2}\int PY\,dT$[1]

[1]This apparent contradiction is inherent in the method of time scaling used. No inconsistencies result if all steps are followed correctly.

The machine equations are now

$$10\ PX + 40X + 8\ PY + 40\ Y = 6$$

$$10\ X + 2\ PY + 12\ Y = 0$$

If the first of these is solved for PY, PX becomes one of the components of PY. There appears to be no way to obtain PX from the second equation without differentiating. Consequently, the first equation is solved for PX, and the second for PY.

$$PX = 0.6 - 4\ X - 0.8\ PY - 4\ Y$$

$$PY = -6Y - 5X$$

In the present problem it is not required to record PX. Hence the machine equations may be written

$$X = \int_0^T (0.6 - 4\ X - 0.8\ PY - 4\ Y)\ dT$$

$$PY = -6\ Y - 5\ X$$

The logic diagram and the implementation used in the solution are shown in Figures 11a and 11b.

Before discussing the solution to the problem there are two points worth mentioning. First, it is noticed that the number of operational amplifiers in any simple loop is always odd. This produces a negative gain around the loop. A positive value (greater than one) of loop gain would cause oscillation.

The other item concerns initial conditions. The initial values of X, Y, and PY are zero, hence it is not necessary to insert initial conditions. Normally the initial value of a variable is inserted by connecting an initial conditions power supply in series with the break relay contact across the integrating capacitor. One may ask,

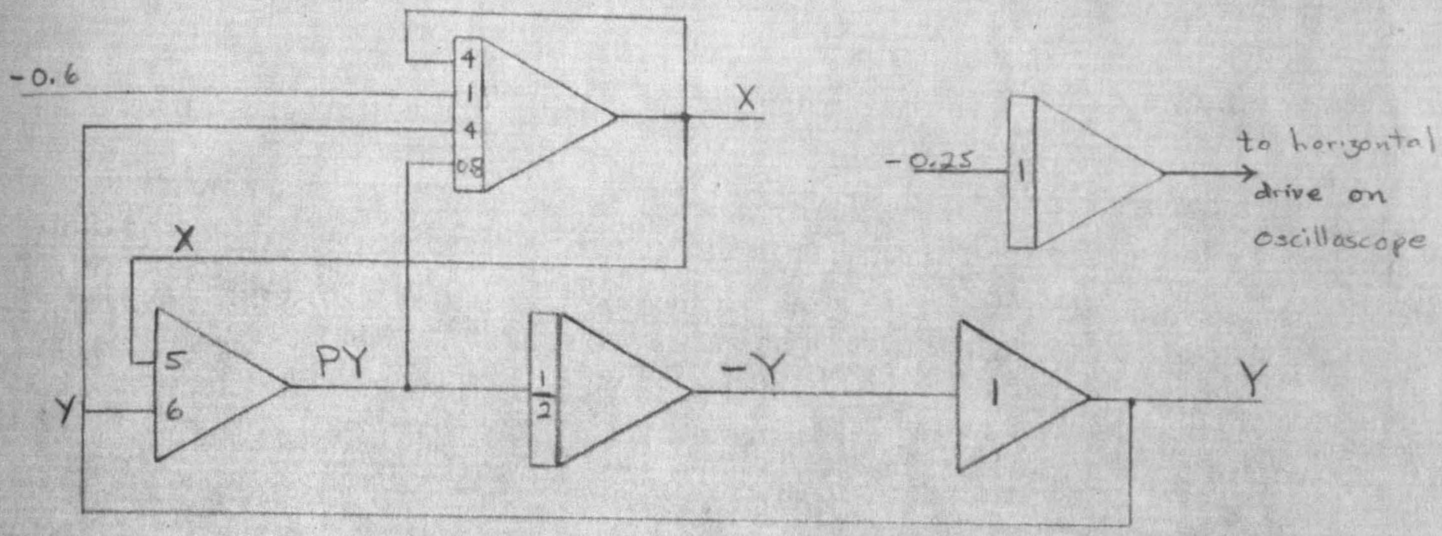

Figure 11a. Logic for Solution of Machine Equations

$$X = \int_0^T (0.6 - 4X - 0.8PY - 4Y)\, dT$$

$$PY = -6Y - 5X$$

$$\text{where } Y = \tfrac{1}{2} \int_0^T PY\, dT$$

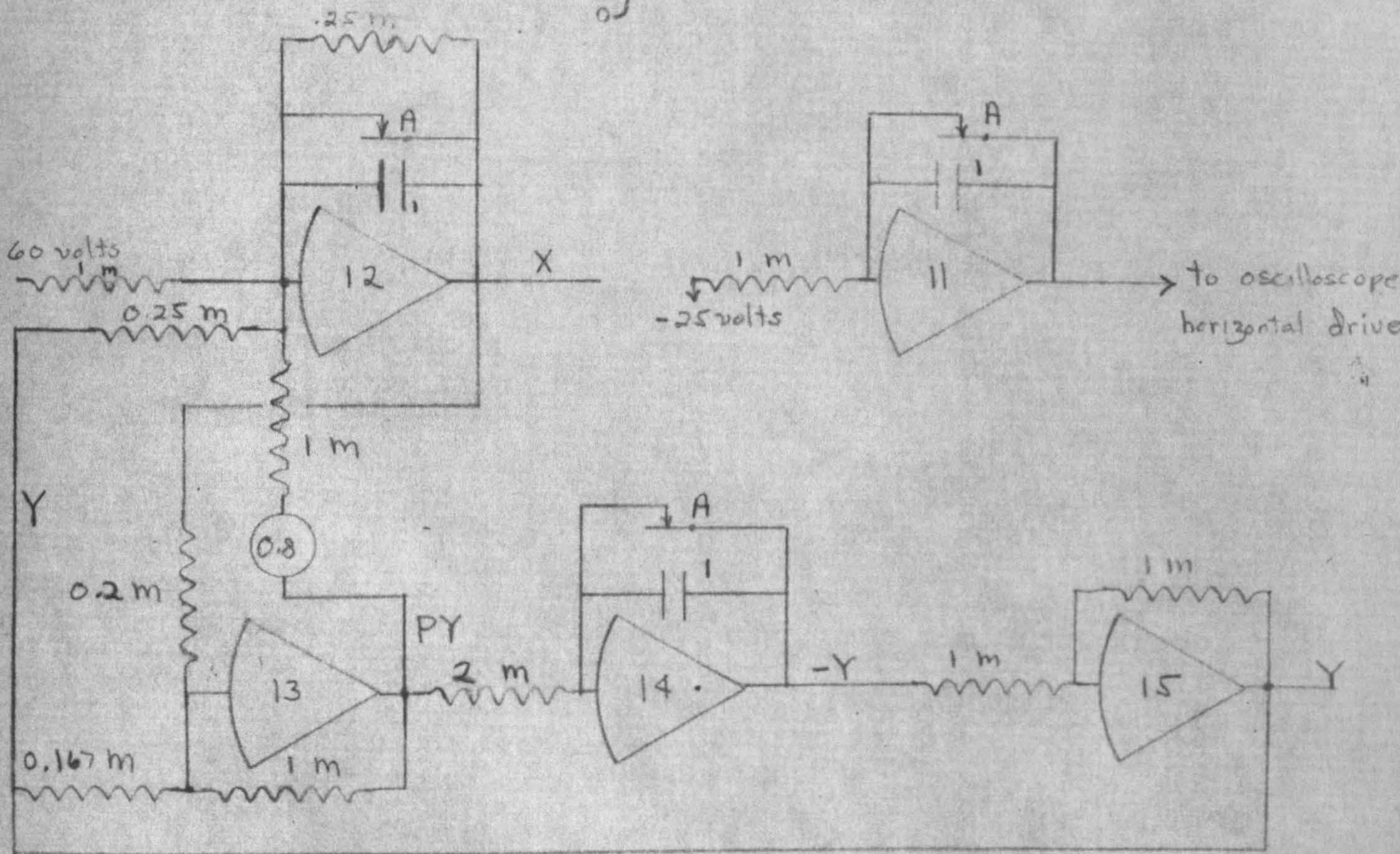

Figure 11b. Implementation of Figure 11a.

however, how initial conditions are set in a variable which appears at the output of an adder where no integrating capacitor is available. The answer is that such an initial condition automatically sets itself. This was confirmed in the present example by using a slightly more elaborate setup (not shown) than shown in Figure 11a. PX was generated and its value measured before the start of the problem. Its measured value was 60 volts or 0.6 machine units which is in agreement with the calculated value.

Figures 11a and 11b show the use of an integrator to drive the scope. The scope was set up so that 0 volts of horizontal drive positioned the beam on the left side of the face, and 100 volts placed it on the right. Applying -25 volts to an integrator which has a gain of 1 results in a linear increasing output which rises to 100 volts in 4 seconds. Hence, the horizontal trace on the scope represents 4 seconds.

The results are shown in Figure 11c. A quick inspection shows that Y and PY are in correct relation in that PY appears to be of the form of the derivative of Y. Further analysis shows that these drawings very closely fit the analytical solutions of the machine equations for both the transient and steady state conditions. The solutions to the original equations are easily obtained by multiplying the values of X by 10, those of Y by 4, and those of PY by 2.

Before going ahead, an observation is in order. The equations under discussion were also implemented in a slightly modified form of Figure 11b. The 0.167 meg resistor was replaced by a potentiometer set at 0.6 (including loading effects) feeding into a 0.1 meg resistor to give the amplifier a gain of 10. The combined gain was

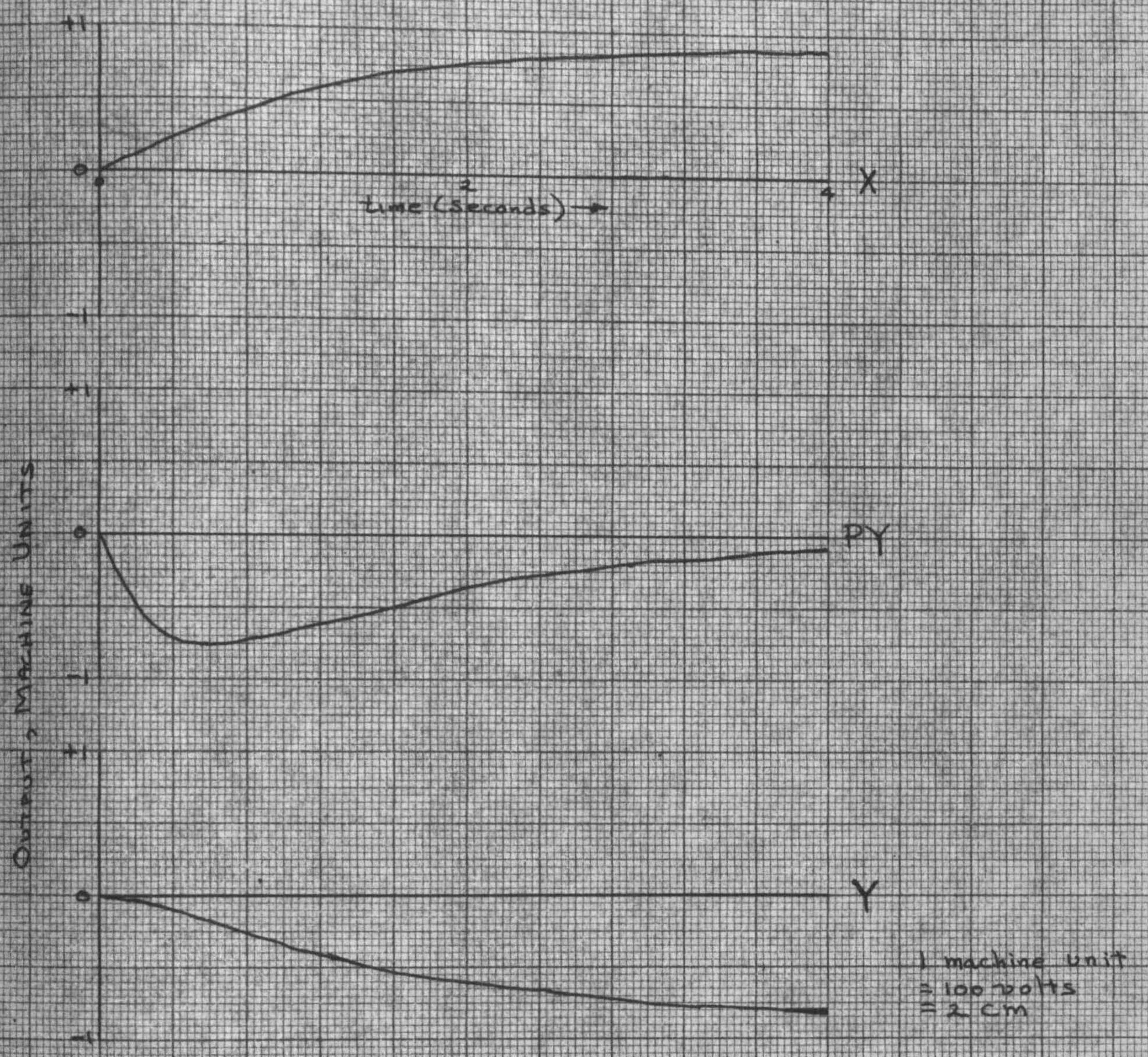

Figure 11c. Oscilloscope Results of Setup of Firure 11b For
Real Time and Fast Time
(Actual Size)

then the required 6. With this setup, the voltage X reached a steady state level of about 0.63 machine units, rather than 90. A small correction in the potentiometer setting to about 0.56 gave the correct results. This result was obtained about four times, running the experiment on different dates with different amplifiers. It was concluded, first of all, that the equation solution was quite sensitive to the voltage at this point. Secondly it appears that one must be extremely cautious when loading potentiometers with values of resistance as low as 0.1 meg. It is better to avoid possible difficulties by using the arrangement shown in Figure 11b.

The same set of equations were also solved at 10 times the speed of the first solutions, or with $dt = 0.1$. For this, the scale factors are:

$$x = 10X$$

$$px = PX = \left(\frac{P}{10}\right)\left(10X\right)$$

$$y = 4Y$$

$$py = 0.2\ PY = \left(\frac{P}{10}\right)\left(2Y\right)$$

$$t = 10T \quad \text{or} \quad p = P/10$$

$$\text{also} \quad Y = \frac{1}{2}\int_0^T PY\,dT$$

Substituting in the original equations:

$$PX + 40\ X + 0.8\ PY + 40\ Y = 6$$

$$10\ X + 0.2\ PY + 12\ Y = 0$$

Solving the first equation for X and the second for $\frac{PY}{10}$:

$$X = 10 \int_0^T (0.6 - 4\,X - 0.08\,PY - 4\,Y)\,dT$$

$$\frac{PY}{10} = -5\,X - 6\,Y$$

The logic diagram for this setup is shown in Figure 12. The figure is identical to Figure 11a except for the fact that the integrator gains have been increased by a factor of 10, and the output of one of the amplifiers is now $\frac{PY}{10}$, rather than PY. Hence, the logic is implemented by simply replacing the 1.0 μF capacitors of Figure 11a by 0.1 μF capacitors. This applies also for the amplifier driving the oscilloscope. The results obtained were identical to those shown in Figure 11c.

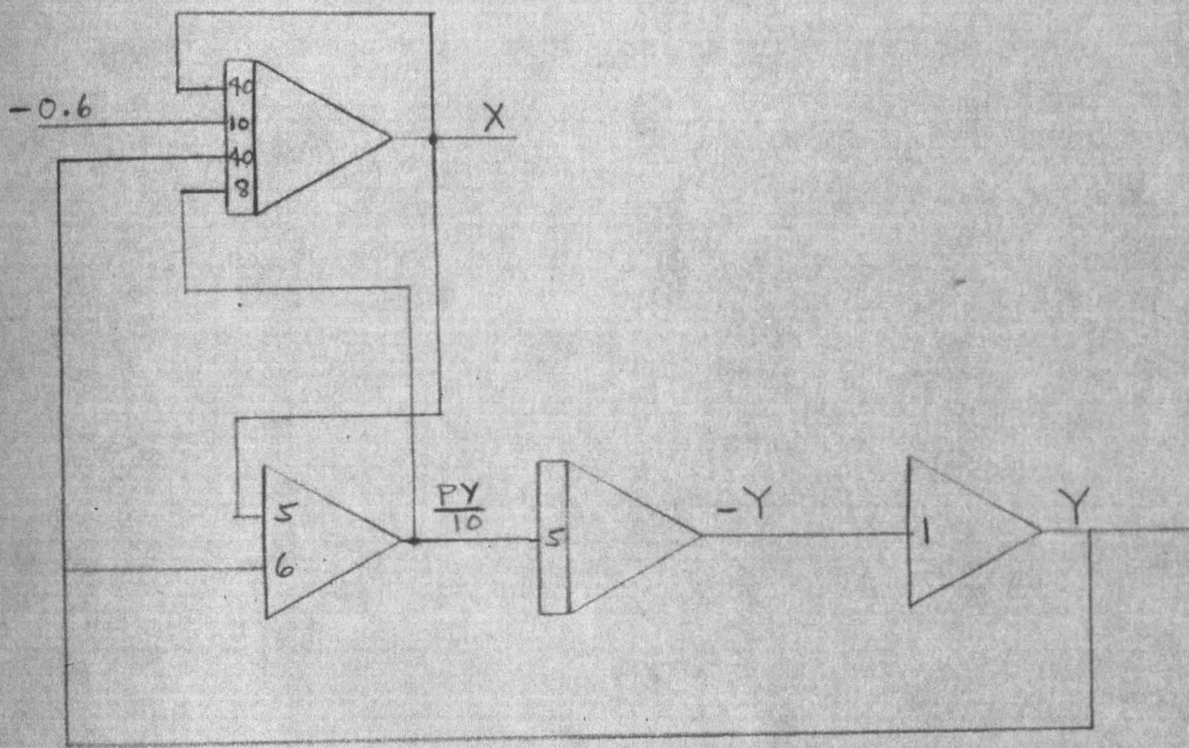

Figure 12. Logic for Solution of Machine Equations

$$X = 10 \int_{0}^{T} (0.6 - 4X - 0.08PY - 4Y)\, dT$$

$$\frac{PY}{10} = -5X - 6Y$$

C. A SIMPLE SERVOMECHANISM

Perhaps the most extensive and fruitful applications of dc analog computers have been in the field of automatic control engineering. Dc analog computing elements lend themselves naturally to the representation of feedback loops analogous to those used in control systems.

Figure 13a[1] shows a simple servomechanism designed to position a load so as to follow the motion of a control dial. A potentiometer type pickoff device measures the output error

$$\epsilon = x_i - x_o$$

and produces a dc error voltage

$$e_1 = a_1\epsilon = a_1(x_i - x_o)$$

which modulates a 60 cps ac carrier. The modulated ac is amplified and controls the torque of a two-phase servomotor so as to reduce the error. The equation of motion of the system is

$$(I_L + n^2 I_m)\, p^2 x_o + n^2 r p x_o = n a_1 a_2 a_3 (x_i - x_o)$$

The identification of the constants and typical values are as follows:

[1]Adapted from Korn and Korn, page 92.

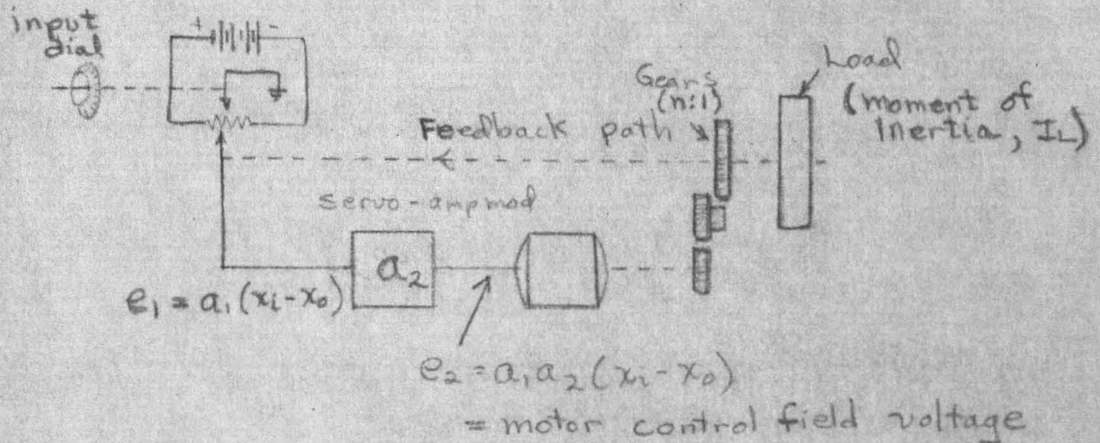

Figure 13a. A Simple Servomechanism Designed to Make the Angular Displacement x_o of a Load Follow the Angular Displacement x_i of an Input Dial

Symbol for Constant	Meaning of Constant	Typical Value of Constant
I_L =	moment of inertia of load	0.05 slug ft^2
I_m =	moment of inertia of motor armature	2×10^{-5} slug ft^2
n =	gear ratio between motor and load	100
r =	motor damping coefficient	10^{-4} (ft-lb-sec)
a_1 =	feedback coefficient	20 volts/radian
a_2 =	amplification factor of servo amplifier-modulator	25
a_3 =	motor stall torque constant	2×10^{-5} ft-lb/volt

It is desired to study the effect on the servo performance of varying the motor damping coefficient r between 0 and 2×10^{-4} ft-lb-sec. Physically this might be done by changing the resistance of the motor armature winding. The response is desired when the input receives a step function of one radian. Two cases are considered, namely starting the problem after an equilibrium has been established, and starting the problem during an unstable period when the output is already one radian behind the input.

Using the above values for the constants, but leaving r as a variable, the equation of motion becomes

$$0.25p^2x_o + 10^4 rpx_o = x_i - x_o$$

or

$$px_o = \int_o^t \left[4(x_i - x_o) - 4 \times 10^4 rpx_o\right] dt$$

The limits of x_o, x_i, and ϵ are not expected to exceed plus or minus 2 radians in this problem, and the limits of px_o not to exceed an absolute value of 8 radians per second. Suitable scale factors are therefore:

$$a_{x_o} = \frac{1}{2} \qquad\qquad t = T$$

$$a_{px_o} = \frac{1}{8}$$

$$a_{x_i} = \frac{1}{2}$$

and the transformation equations are

$$x_o = 2X_o$$

$$px_o = 8PX_o$$

$$x_i = 2X_i$$

with $X_o = 4 \int_o^T PX_o dT$

Thus the machine equation is

$$PX_o = \int_o^T \left[(x_i - x_o) - 4 \times 10^4 rPX_o\right] dT$$

with $X_o = 4 \int_o^T PX_o dT$

Figures 13b and 13c show the logic and implementation of the machine equation. The input was connected to an initial conditions supply of -50 volts. The response (X_o) was obtained for zero initial conditions and for an initial displacement of minus one radian by using an initial conditions voltage of 50 volts across amplifier 13. The results are shown in Figure 13d for the two different initial conditions and for five values of r. As expected, an increase in the motor damping coefficient damps out the oscillations. Increasing the coefficient beyond the critical damping point increases the time required for the transient to die out. The output may be expressed in radians by doubling the number of machine units shown in Figure 13d.

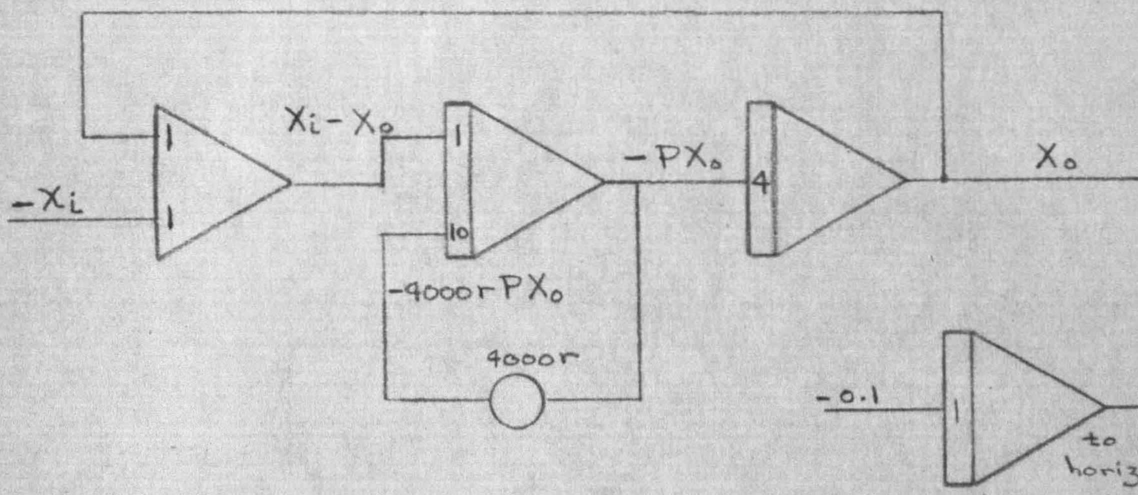

Figure 13b. Logic for Solution of Machine equation

$$PX_o = \int_0^T \left[(X_i - X_o) - 4 \times 10^4 rPX_o \right] dT$$

$$\text{with} \quad X_o = 4 \int_0^T PX_o dT$$

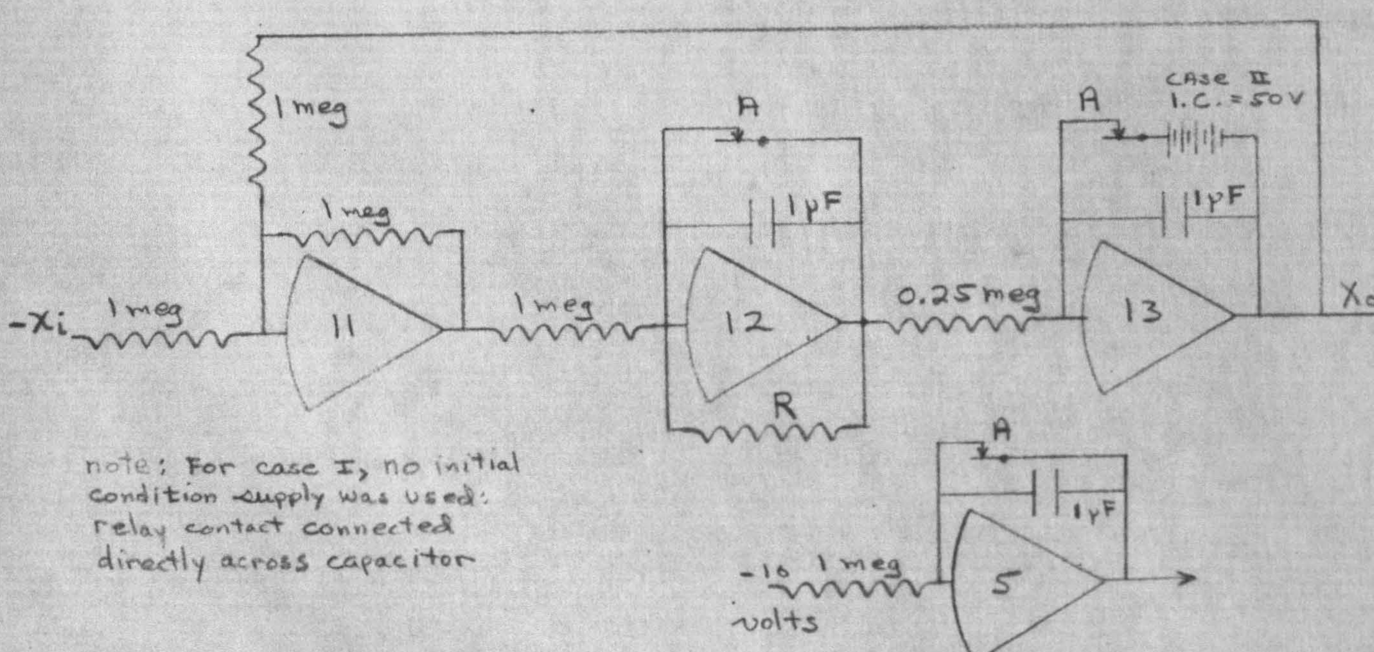

Figure 13c. Implementation of Figure 13b.

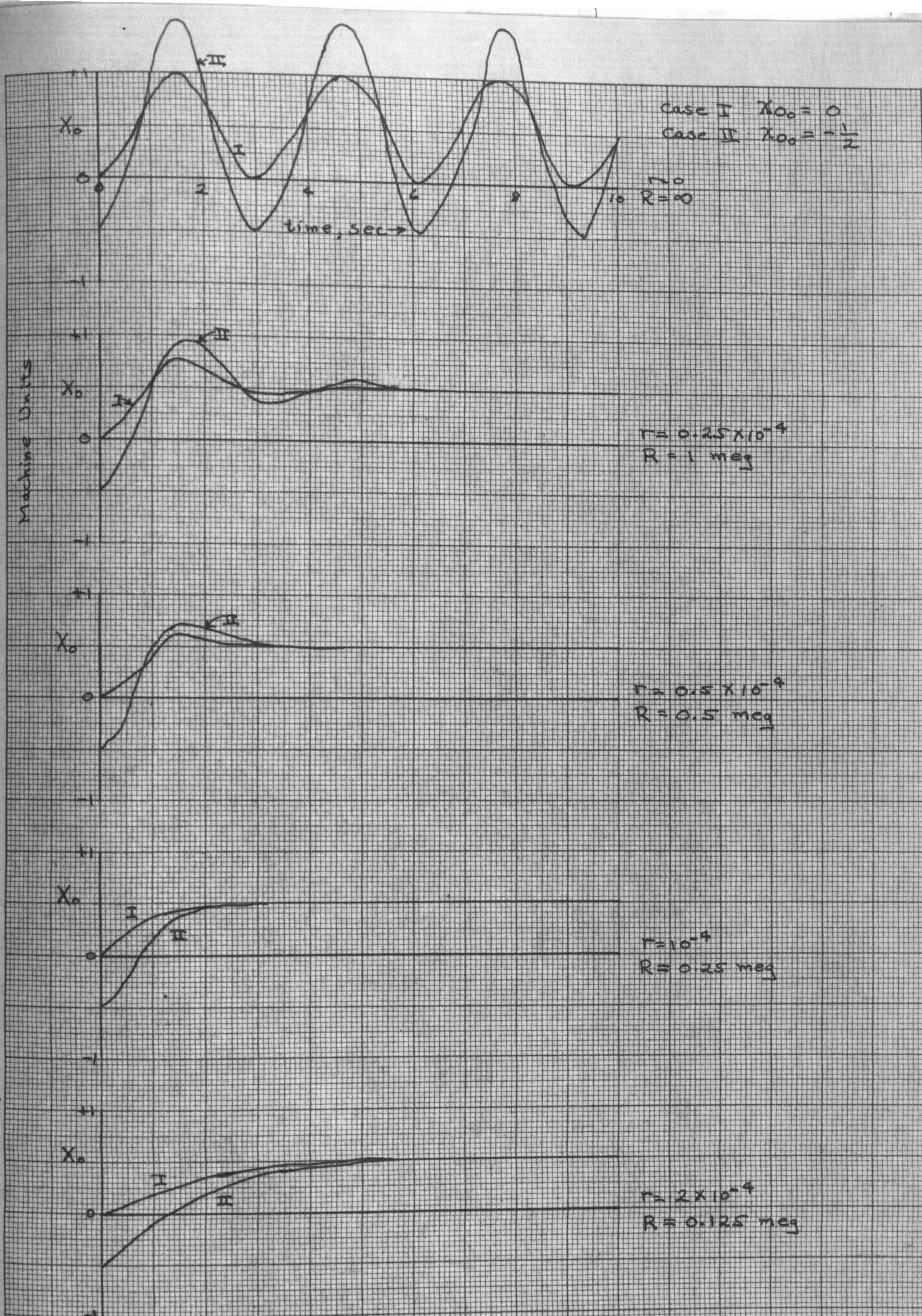

Figure 13d. Oscilloscope Observations of Operation of Figure 13c.

It is noted in Figure 13d that in one case the output exceeded one machine unit in amplitude. Although it is not good practice to design for such values of output, the operational amplifier was apparently able to produce the required output without clipping.

One final observation is in order. The equation of motion may be rewritten:

$$X_o = \frac{X_i}{0.25\, p^2 + 10^4\, rp + 1}$$

The final value theorem may be applied to this equation to determine the steady state response of the system. To do this, X_i, which is to be a step input of one radian, is replaced in the above equation by $1/p$. Then:

$$\lim_{t \to \infty} X_o = \lim_{p \to 0} p \left(\frac{1/p}{0.25p^2 + 10^4\, rp + 1} \right)$$

Clearly, this value is one radian; the result obtained by the simulation.

D. A FREQUENCY ANALYZER

The last example is not one of a problem that has been run on the computer; rather, it is a qualitative analysis to illustrate an idea.

Suppose one considers the circuit of Figure 14a. For frequencies other than resonance, there will be little voltage across the tuned circuit. If, however, either the generator frequency is varied or the tuned circuit is adjusted to produce resonance, a substantially large voltage is developed across the tuned circuit. This voltage is rectified and filtered to produce an output on the meter. Such a device is useful for determining the frequencies of the various components of unknown signal. It is interesting to consider replacing the tuned circuit with components of an analog computer.

The part of the circuit of interest is redrawn in Figure 14b. The following loop equations apply to the circuit.

$$e_i = (R + \frac{1}{pC})\ i_1 - (\frac{1}{pC})\ i_2$$

$$0 = -(\frac{1}{pC})\ i_1 + (pL + \frac{1}{pC})\ i_2$$

Also:

$$e_o = e_i - i_1 R$$

These equations may be solved to give the transform

$$\frac{e_o}{e_i} = \frac{1}{1 + RpC + R/pL}$$

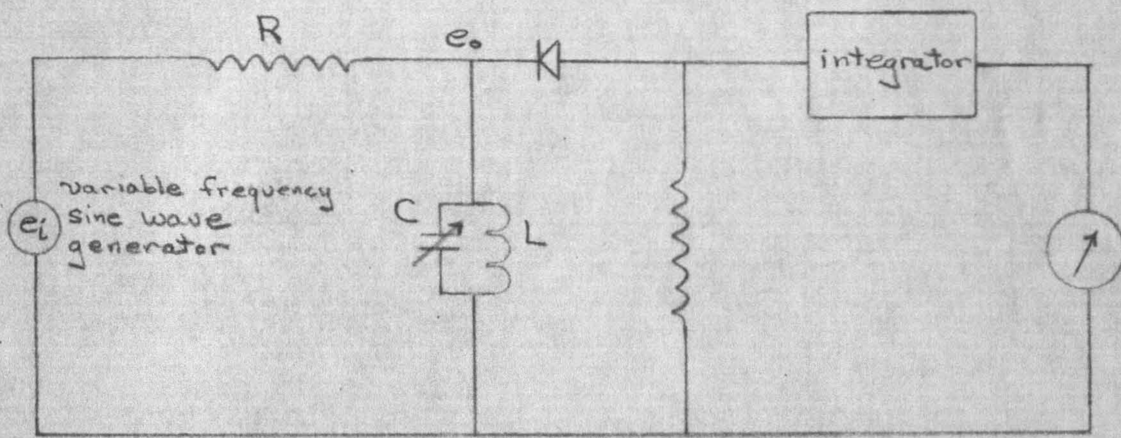

Figure 14a. A Device for Determining the Frequency Components of an Unknown Input Signal.

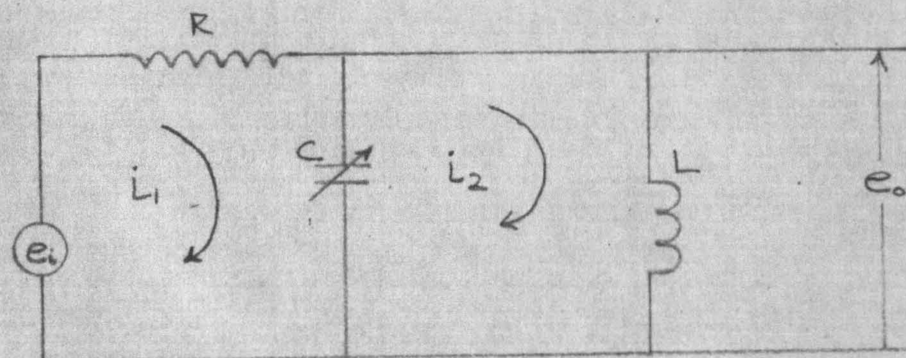

Figure 14b. A Partial Redrawing of Figure 14a Showing Loop Currents

Therefore

$$pe_o = \frac{e_i}{RC} - \frac{e_o}{RC} - \frac{e_o}{pLC}$$

and

$$e_o = \int_o^t \left(\frac{e_i}{RC} - \frac{e_o}{RC} - \frac{e_o}{pLC} \right) dt$$

This equation may be implemented with the logic of Figure 14c. A signal of unknown frequency components can be connected to the input, and the output can be rectified, integrated, and read on a meter or oscilloscope. By noting the values of potentiometer settings at which outputs occur, the values of input frequency are determined. Before actually setting up this experiment, it would of course be necessary to scale the problem for the machine.

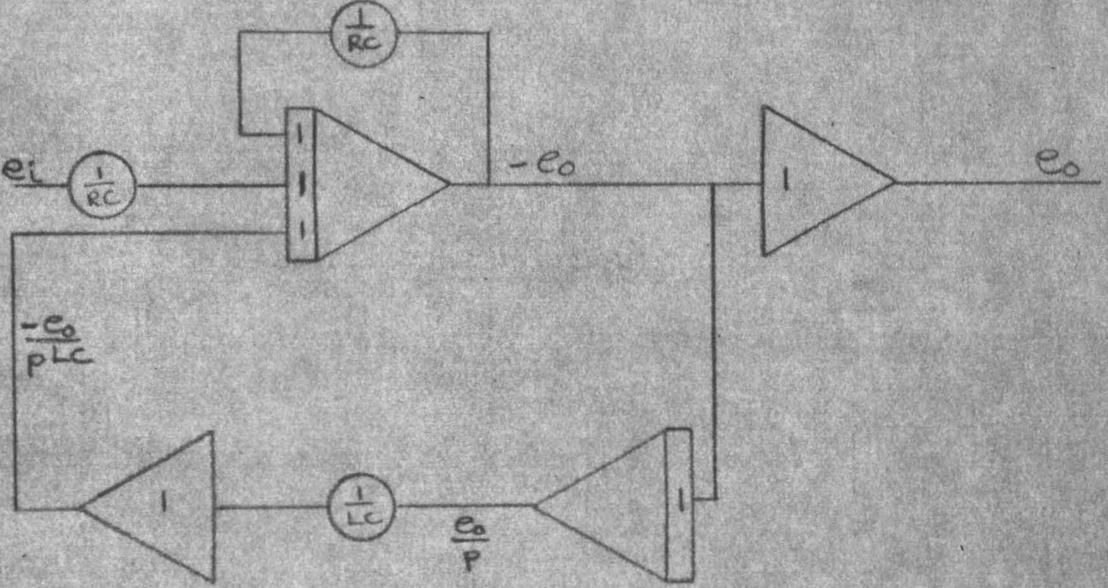

Figure 14c. Logic for Solution of Equation

$$e_o = \int_0^t \left(\frac{e_i}{RC} - \frac{e_o}{RC} - \frac{e_o}{pLC} \right) dt$$

CHAPTER V

SPECIAL CIRCUITS FOR ELECTRONIC ANALOG COMPUTERS

Many special purpose circuits are used in analog computers. A brief mention of several of these follows.

A. MULTIPLIERS

Multiplication is usually accomplished on an analog computer by the use of servomultipliers. The Heath Computer does not contain servomultipliers, perhaps partly in the interests of simplicity and partly due to their inability to follow at the frequencies of the repetitive mode of the computer. As an alternative, electronic methods for multiplication are feasible. Although multiplication was not tried on the Heath Computer, an attempt is made here to discuss some of the methods which might be considered practicable.

The first method involves a time-division multiplier[1] and is illustrated in Figure 15. The positive voltage $Ey = Y + X_2$ charges the integrating capacitor until the voltage E_1 reaches a value sufficient to energize the relay. The integrating network then integrates $Ey = -Y + X_2$ so that E_1 decreases until the relay releases. The entire process is then repeated. The mathematics of the situation show that the output X_o is proportional to the product X_1X_2, but inversely proportional to Y. Since Y can be fixed,

[1]Korn and Korn, page 270.

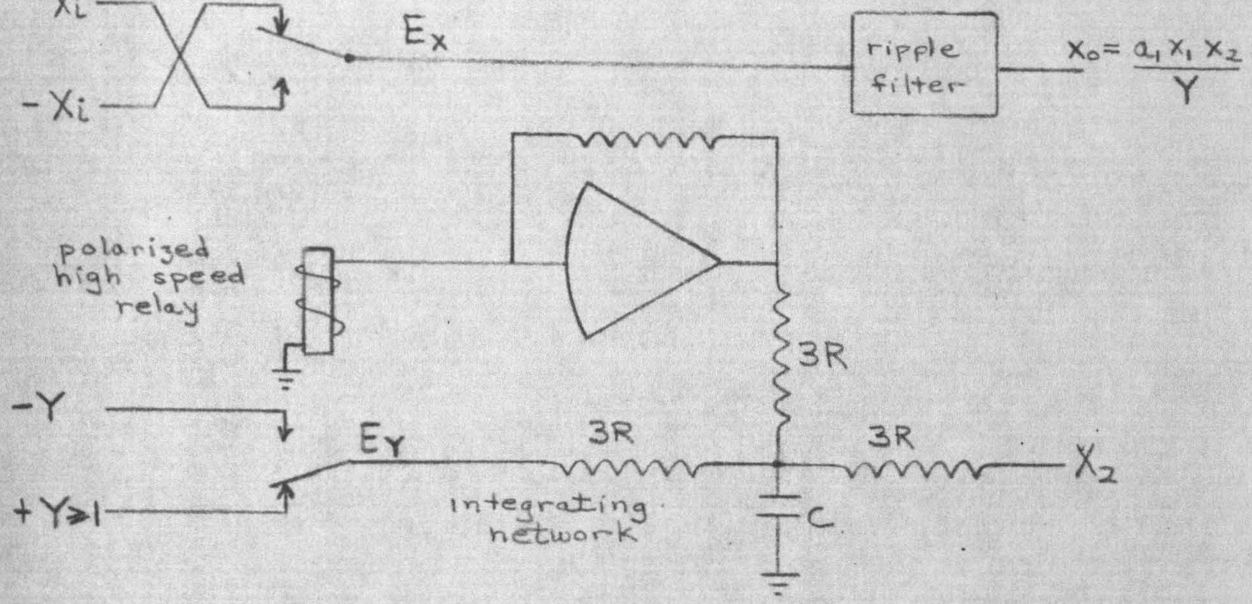

Figure 15. Relay Time Division Multiplier Circuit for Four Quadrant Operation.

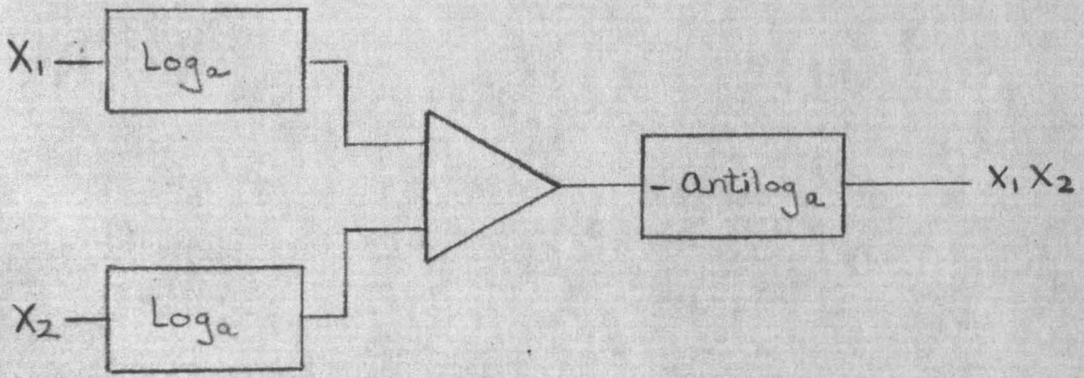

Figure 16. A Logarithmic Multiplier

the device can be used to multiply the quantities X_1 and X_2. This device is suitable for four-quadrant operation, meaning that it will accept both positive and negative values of the input variables.

Another type of multiplier is illustrated in Figure 16[1]. It is based on the relationship

$$X_1X_2 = a^{(\log_a X_1 + \log_a X_2)} = \text{antilog}_a\ (\log_a X_1 + \log_a X_2)$$

The log generators are nonlinear function generators which can be synthesized from special diode circuits. The antilogarithm function can be accomplished by generating the inverse function of the logarithm. This inverse function, or in general, any inverse function can be generated by putting the forward function generator in the feedback of an operational amplifier[2]. It is to be noted that the direct application of Figure 16 permits only positive values of X_1 and X_2 (one quadrant operation). A multiplier of this type is included in the Donner Model 30 analog computer.

One very ingenious method of multiplying is the amplitude selecting multiplier provided in the Philbrick Analog Computer[3]. The input voltages are combined with a high frequency triangular wave voltage. Sums and selections of the greater of two voltages

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[1]Korn and Korn, page 281.
[2]Korn and Korn, page 340.
[3]Korn and Korn, pages 282 and 409.

are carried out in such a manner that the final output is proportional to the product of the inputs.

B. DELAY OR DEAD TIME

A great many real situations involve time delays, that is, delays in outputs of components and systems, once an input signal has been given. If such a system is to be simulated with analog components, it is important that the delays also be simulated. A brief description of the theory of time delay simulation is as follows.

An important theorem of Laplace transforms states that

$$\mathcal{L}\left[f(t - \tau)\right] = e^{-\tau p}\,\mathcal{L}\left[f(t)\right]$$

and so the transfer function of dead space is

$$\frac{E_o}{E_i} = e^{-\tau p}$$

This may be replaced by the somewhat primitive approximation $1/(1 + \tau P)$ or by any of a series of successively better approximations. These in turn can be implemented by analog computer components.

C. DEAD SPACE

In dead space circuits, there is no output for a particular range of the input, but output does occur for input values above and below this range. The situation is encountered in real life in many mechanical and electrical transducers such as tachometers, accelerometers, gyroscopes and pressure gauges. The phenomenon is characterized by a region of no sensitivity, i.e., an inert zone, or dead space. Diode circuits or special relay circuits may be used to simulate this condition.

D. HYSTERESIS AND BACKLASH

Hysteresis is the phenomenon exhibited by a system whose state depends upon its previous history. Such effects occur in magnetic materials and in gear chains with backlash. Such effects may be simulated on an analog computer by a combination of dead space and memory, the memory being provided by a capacitive store.

CHAPTER VI

CONCLUSION

At the present stage of engineering progress there appears to be a trend toward digital rather than analog methods of computation. Digital computers seem better suited than analog computers for processing the tremendously large amounts of data that are the inputs to many present day problems. Examples of such problems include payroll calculations, inventory control, and calculation of trajectory intersections with radar beams in military applications.

In spite of the above tendency, the author believes that analog methods of computation should be a part of every good engineer's "vocabulary." Analog methods seem to possess a certain beauty in their methods of direct simulation of physical systems and differential equations, in contrast to the methods used in digital computers where numerical methods of solving such equations must first be specified. The study of an analog computer setup gives insight into the meaning of a differential equation and its solution.

The author believes that the Heath Analog Computer will fulfill a primary need not in industry, but rather as an educational tool. It appears to be excellent as an introduction to analog computers and is within a price that can be afforded by many educational institutions. This does not mean that the Heath computer must be excluded from commercial applications completely. In general though, industrial type analog computers are usually larger, more accurate, and more convenient to use than the Heath computer. Many industrial problems require a computer with more than fifteen operational amplifiers. Even here, however, several Heath computers could be used together, or the Heath could act as a slave or master of another computer. Many industrial computers use chopper stabilized amplifiers and temperature controlled components in order to attain high accuracy. Industrial computers also incorporate features such as predetermined integrators and multipliers, push buttom potentiometer setting, and removable patch boards. This last feature permits one to set up a problem while someone else is using the computer. It also permits retention of a setup which might be required from time to time alternately with other problems that are worked on the computer. Not to be forgotten also is the fact that many industrial computers contain convenient devices for multiplication of one variable by another.

In the interests of simplicity and cost considerations, the Heath computer has omitted the above mentioned features. Rather, it serves its purpose in acquainting the student with analog methods, and the above features are unnecessary for this purpose.

As a final word, let it be said that the primary purpose served by the writing of this thesis was to acquaint the author with the analog computer and analog methods of computation. It is hoped that the results of this work will also serve as an introduction and guide to other students who may wish to pursue this subject.

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