Difference Equations to Differential Equations

Section 1.2

Sequences

Recall that a sequence is a list of numbers, such as

$$1, 2, 3, 4, \ldots,$$
$$2, 4, 6, 8, \ldots,$$
$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots,$$
$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots,$$

or

$$1, -1, 1, -1, \ldots.$$

As we noted in Section 1.1, listing the first few terms of a sequence does not uniquely specify the remaining terms of the sequence. To fully specify a sequence, we need a formula that describes an arbitrary term in the sequence. For example, the first example above lists the first four terms of the sequence $\{a_n\}$ with

$$a_n = n$$

for $n = 1, 2, 3, \ldots$; the second example lists the first four terms of $\{b_n\}$ with

$$b_n = 2n$$

for $n = 1, 2, 3, \ldots$; the third example lists the first four terms of $\{c_n\}$ with

$$c_n = 1 - \frac{1}{n}$$

for $n = 1, 2, 3, \ldots$; the fourth lists the first four terms of $\{d_n\}$ with

$$d_n = \frac{(-1)^n}{2^n}$$

for $n = 0, 1, 2, 3, \ldots$; and the fifth lists the first four terms of $\{e_n\}$ with

$$e_n = (-1)^n$$

for $n = 0, 1, 2, \ldots$.

As indicated in Section 1.1, we are often interested in the value, if one exists, which a sequence approaches. For example, the sequences $\{a_n\}$ and $\{b_n\}$ increase beyond any possible bound as n increases, and hence they have no limiting value. To visualize what is happening here, you might plot the points of the sequence on the real line. For both of these sequences, the plotted points will march off to the right without any upper limit. Although a limit does not exist in these cases, we usually write

$$\lim_{n\to\infty} a_n = \infty$$

and

$$\lim_{n\to\infty} b_n = \infty$$

to express the fact that the limits do not exist because the terms in the sequence are growing without any positive bound. On the other hand, if we plot the points of the sequence $\{c_n\}$, as in Figure 1.2.1, we see that although they are always increasing (that is, moving toward the right), nevertheless they never increase beyond 1. Moreover, even though no term in the sequence is ever equal to 1, we can see that the points become arbitrarily close to 1. Hence we say that the limit of the sequence is 1 and we write

$$\lim_{n\to\infty} c_n = 1.$$

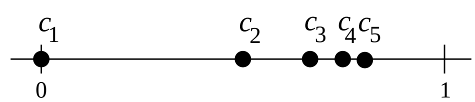

Figure 1.2.1 The first five values of $c_n = 1 - \frac{1}{n}$

Even though they oscillate between positive and negative values, the terms in the sequence $\{d_n\}$ approach closer and closer to 0 as n increases. Since it is possible to make d_n as close as we like to 0 by taking n suitably large, we may write

$$\lim_{n\to\infty} d_n = 0.$$

Finally, for the sequence $\{e_n\}$ there are only two points to plot, alternating between 1 and -1. Since the terms of this sequence oscillate between two numbers, and so do not approach any fixed limiting value, we say that the sequence does not have a limit.

Another approach to visualizing the limiting behavior of a sequence $\{a_n\}$ is to plot the ordered pairs (n, a_n) in the plane for some range of values of n. For example, Figure 1.2.2 shows a plot of the points (n, c_n), $n = 1, 2, 3, \ldots, 50$ for the sequence $\{c_n\}$ given above. Note how the points approach the horizontal line $y = 1$, indicating, as mentioned above, that

$$\lim_{n\to\infty} c_n = 1.$$

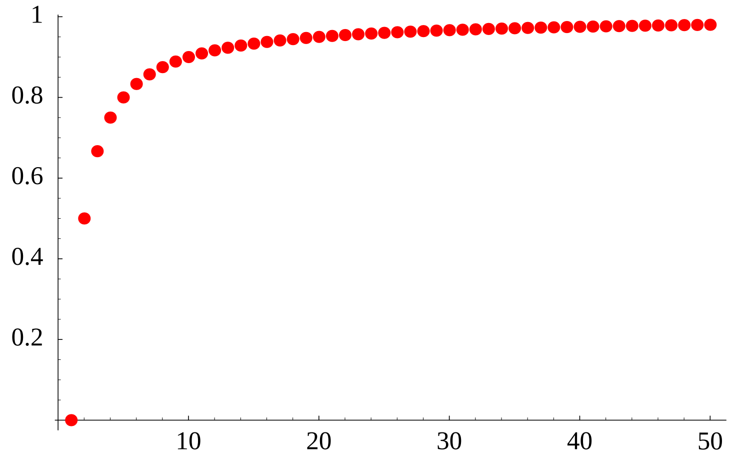

Figure 1.2.2 Plot of $(n, 1-\frac{1}{n})$ for $n = 1, 2, 3, \ldots, 50$

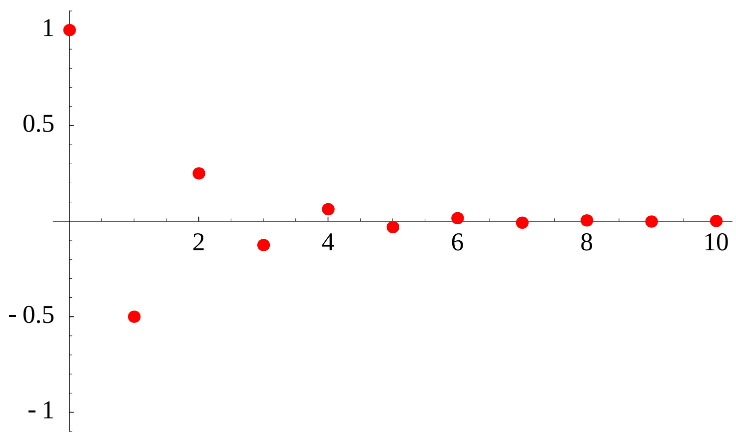

Figure 1.2.3 Plot of $(n, \frac{(-1)^n}{2^n})$ for $n = 0, 1, 2, \ldots, 10$

Similarly, Figure 1.2.3 shows a plot of the points (n, d_n), $n = 0, 1, 2, \ldots, 10$; here the points approach the horizontal axis, $y = 0$, consistent with our claim that

$$\lim_{n\to\infty} d_n = 0.$$

Figure 1.2.4 shows a plot of (n, e_n), $n = 0, 1, 2, \ldots, 20$. The fact that this sequence does not have a limit is manifest in seeing the vertical coordinate of the points oscillate between 1 and -1.

As the concept of a limit is fundamental to the understanding of calculus, it is important that we make the notion more concrete than we have so far. That is, we need to have a formal definition of limit which exactly captures what we have been discussing intuitively. The idea is that we should say L is the limit of a sequence $\{a_n\}$ if for any open interval I containing L, no matter how small, we can find a point in the sequence beyond which all values of the sequence lie in I. Graphically, this means that if we start plotting the points of the sequence, there will come a time when all points from then on will lie

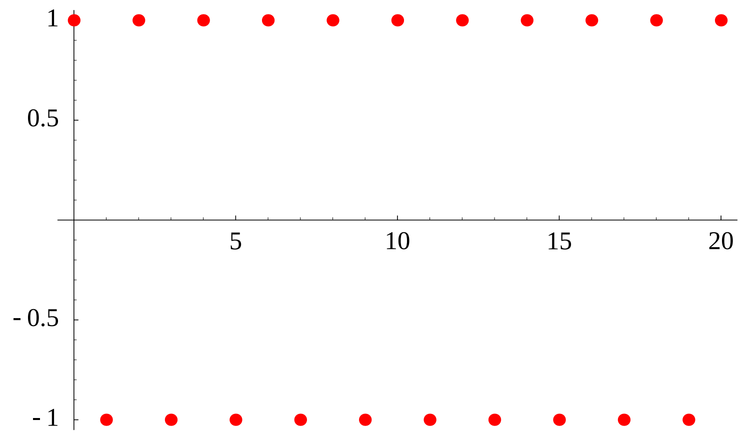

Figure 1.2.4 Plot of $(n, (-1)^n)$ for $n = 0, 1, 2, \ldots, 20$

within the interval I. This idea is formalized in the following definition, where the open interval I is expressed in the form $(L-\epsilon, L+\epsilon)$ and the idea that all values of the sequence beyond a certain point are in this interval is expressed by requiring that $|a_n - L| < \epsilon$, that is, the distance between a_n and L is less than ϵ, for all $n > N$.

Definition We say that the *limit* of the sequence $\{a_n\}$ is L, written

$$\lim_{n\to\infty} a_n = L,$$

if for every $\epsilon > 0$ there exists an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

Hence to show that the limit of a sequence is a number L, one must show that for any positive number ϵ, it is possible to find an integer N such that the numbers $a_{N+1}, a_{N+2}, a_{N+3}, \ldots$ are all in the interval $(L-\epsilon, L+\epsilon)$. See Figure 1.2.5.

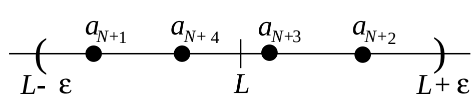

Figure 1.2.5 a_n in $(L-\epsilon, L+\epsilon)$ for $n > N$

Example We will show that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

To do so, we must show that for any given $\epsilon > 0$, we can find an integer N such that

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

whenever $n > N$. Now

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n},$$

so we need only determine the values of n for which

$$\frac{1}{n} < \epsilon.$$

Since

$$\frac{1}{n} < \epsilon \text{ if and only if } n > \frac{1}{\epsilon},$$

it follows that we may take N to be the largest integer less than or equal to $\frac{1}{\epsilon}$. Then whenever $n > N$, we have

$$n > \frac{1}{\epsilon},$$

from which it follows that

$$\frac{1}{n} < \epsilon.$$

This is exactly what we need in order to conclude, by the definition, that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

The following definition is useful in situations, such as in the previous example, when we want the largest integer less than or equal to some given value.

Definition For any real number x, we may define the *floor function*, denoted $\lfloor x \rfloor$, by

$$\lfloor x \rfloor = \text{ the largest integer less than or equal to } x, \tag{1.2.1}$$

and the *ceiling function*, denoted $\lceil x \rceil$, by

$$\lceil x \rceil = \text{ the smallest integer greater than or equal to } x. \tag{1.2.2}$$

For example, $\lfloor 5.3 \rfloor = 5$, $\lceil \pi \rceil = 4$, $\lfloor 3 \rfloor = 3$, and $\lceil 3 \rceil = 3$. With this notation, we could define N in the previous example by

$$N = \left\lfloor \frac{1}{\epsilon} \right\rfloor.$$

Example We will show that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

This time we must show that for any $\epsilon > 0$, we can find an integer N such that

$$\left| \frac{1}{2^n} - 0 \right| < \epsilon$$

whenever $n > N$. Now

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$

so we need to determine the values of n for which

$$\left(\frac{1}{2}\right)^n < \epsilon.$$

We need to solve this inequality for n. Since n is in the exponent, we may use logarithms to simplify the inequality. Although we will not provide a careful treatment of logarithms until Chapter 6, we will assume for the moment some acquaintance with logarithms using base 10. Now

$$\left(\frac{1}{2}\right)^n < \epsilon$$

if and only if

$$\log_{10}\left(\frac{1}{2}\right)^n < \log_{10}(\epsilon).$$

Since

$$\log_{10}\left(\frac{1}{2}\right)^n = n\log_{10}\left(\frac{1}{2}\right),$$

we have

$$\left(\frac{1}{2}\right)^n < \epsilon$$

if and only if

$$n\log_{10}\left(\frac{1}{2}\right) < \log_{10}(\epsilon).$$

Now $\log_{10}\left(\frac{1}{2}\right) < 0$, so

$$n\log_{10}\left(\frac{1}{2}\right) < \log_{10}(\epsilon)$$

if and only if

$$n > \frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)}.$$

Thus if we let

$$N = \left\lfloor \frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)} \right\rfloor,$$

then

$$\left|\frac{1}{2^n} - 0\right| < \epsilon$$

whenever $n > N$. For example, if we take $\epsilon = 0.001$, then, to two decimal places,

$$\frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)} = 9.97,$$

and so we would have

$$N = \lfloor 9.97 \rfloor = 9.$$

This N works because, for $n > 9$,

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} \leq \frac{1}{2^{10}} = \frac{1}{1024} < 0.001.$$

Problem 12 at the end of this section will ask you to generalize the previous example to show that

$$\lim_{n\to\infty} r^n = 0$$

whenever $|r| < 1$. This is an important fact that we will make use of later.

In this course we will be concerned more with the development of an intuitive understanding of limits and a computational facility with limits than with the formalism of verifying a specific limit using the above definition. That is not to say that the definition is unimportant; rather a good grasp of the concept in the definition is important for a full understanding of much of what we will do in calculus. In fact, mathematicians of the 19th century arrived at the definition we have stated in their attempts to clarify confusions that had developed in mathematics since the time of Newton and Leibniz. However, for the most part these difficulties are beyond the scope of a text such as this one.

We will see that a few basic properties of limits, combined with a few simple limits like the ones in the previous two examples, will enable us to compute easily a large number of limits. To begin considering these properties, consider the case where we already know that

$$\lim_{n\to\infty} a_n = L \tag{1.2.3}$$

and we want to compute

$$\lim_{n\to\infty} ka_n$$

for some constant $k \neq 0$. Now (1.2.3) tells us that for any $\epsilon > 0$, we may find an integer N such that for $n > N$,

$$|a_n - L| < \frac{\epsilon}{|k|}.$$

It follows that for $n > N$,

$$|ka_n - kL| = |k||a_n - L| < |k|\frac{\epsilon}{|k|} = \epsilon.$$

But this is what it means to say that

$$\lim_{n\to\infty} ka_n = kL. \tag{1.2.4}$$

Note that (1.2.4) is obviously true as well when $k = 0$. Hence we have the following proposition.

Proposition If $\{a_n\}$ is a sequence for which

$$\lim_{n\to\infty} a_n = L,$$

then for any constant k we have

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL. \tag{1.2.5}$$

Example Since we have already seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows that

$$\lim_{n\to\infty} \frac{350}{n} = 350 \lim_{n\to\infty} \frac{1}{n} = (350)(0) = 0.$$

Now suppose we have two sequences $\{a_n\}$ and $\{b_n\}$ with

$$\lim_{n\to\infty} a_n = L \tag{1.2.6}$$

and

$$\lim_{n\to\infty} b_n = M. \tag{1.2.7}$$

Then (1.2.6) and (1.2.7) tell us that for any $\epsilon > 0$, we can find integers N_1 and N_2 such that

$$|a_n - L| < \frac{\epsilon}{2}$$

whenever $n > N_1$ and

$$|b_n - M| < \frac{\epsilon}{2}$$

whenever $n > N_2$. If we let N be the larger of N_1 and N_2, then whenever $n > N$ we will have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{1.2.8}$$

Note that in (1.2.8) we have used the fact, known as the *triangle inequality*, that for any real numbers x and y,

$$|x + y| \leq |x| + |y|. \tag{1.2.9}$$

Thus we have shown

$$\lim_{n\to\infty} (a_n + b_n) = L + M. \tag{1.2.10}$$

Hence we have the following proposition.

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M. \tag{1.2.11}$$

Example We have

$$\lim_{n\to\infty}\left(4 + \frac{8}{n}\right) = \lim_{n\to\infty} 4 + \lim_{n\to\infty}\frac{8}{n} = 4 + 8\lim_{n\to\infty}\frac{1}{n} = 4 + (8)(0) = 4.$$

Note that in the last example we used the fact that if k is a constant and $a_n = k$ for all n, then

$$\lim_{n\to\infty} a_n = k.$$

This follows immediately from the definition since

$$|a_n - k| = 0$$

for all values of k, and so any integer N will work for any value of ϵ.

Again suppose we have two sequences $\{a_n\}$ and $\{b_n\}$ with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M.$$

Then we have

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty}(-b_n) = \lim_{n\to\infty} a_n + (-1)\lim_{n\to\infty} b_n = L - M. \qquad (1.2.12).$$

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M. \tag{1.2.13}$$

Example We have

$$\lim_{n\to\infty}\left(\frac{3}{n}-\frac{8}{5^n}\right)=3\lim_{n\to\infty}\frac{1}{n}-8\lim_{n\to\infty}\left(\frac{1}{5}\right)^n=(3)(0)-(8)(0)=0.$$

Note that we have used the result that

$$\lim_{n\to\infty} r^n=0$$

whenever $|r|<0$.

We will state three more properties of limits without justifications. Although the reasoning behind these results is similar to the reasoning of the previous three propositions, they require a little more care and are best left to a more advanced course.

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n=L$$

and

$$\lim_{n\to\infty} b_n=M,$$

then

$$\lim_{n\to\infty} a_nb_n=(\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)=LM. \tag{1.2.14}$$

Example We have

$$\lim_{n\to\infty}\frac{1}{n^2}=\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)=(0)(0)=0.$$

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n=L$$

and

$$\lim_{n\to\infty} b_n=M,$$

then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}=\frac{L}{M}, \tag{1.2.15}$$

provided $L\neq 0$ and $b_n\neq 0$ for all n.

Example We have

$$\lim_{n\to\infty}\frac{n-3}{2n+4}=\lim_{n\to\infty}\frac{\dfrac{n-3}{n}}{\dfrac{2n+4}{n}}=\lim_{n\to\infty}\frac{1-\dfrac{3}{n}}{2+\dfrac{4}{n}}=\frac{\lim\limits_{n\to\infty}\left(1-\dfrac{3}{n}\right)}{\lim\limits_{n\to\infty}\left(2+\dfrac{4}{n}\right)}=\frac{1}{2}.$$

Note that we can apply the previous proposition only when both numerator and denominator have a limit. Hence, in this example, we first divided the numerator and denominator by n to put the problem in a form to which we could apply the proposition.

Proposition Suppose $\{a_n\}$ is a sequence with

$$\lim_{n\to\infty} a_n = L.$$

Moreover, suppose p is a rational number, a_n^p is defined for all n, and L^p is defined. Then

$$\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p = L^p. \tag{1.2.16}$$

Example We have

$$\lim_{n\to\infty} \sqrt{4 - \frac{3}{n}} = \sqrt{\lim_{n\to\infty} \left(4 - \frac{3}{n}\right)} = \sqrt{4} = 2.$$

Example For any rational number $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^p} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^p = 0^p = 0.$$

Example We have

$$\lim_{n\to\infty} \left(18 - \frac{5}{n} + \frac{23}{n^5}\right) = \lim_{n\to\infty} 18 - 5 \lim_{n\to\infty} \frac{1}{n} + 23 \lim_{n\to\infty} \frac{1}{n^5} = 18 - (5)(0) + (23)(0) = 18.$$

Example We have

$$\lim_{n\to\infty} \frac{4n^5 + 5n^2 - 6}{3n^5 + 4n - 18} = \lim_{n\to\infty} \frac{4 + \dfrac{5}{n^3} - \dfrac{6}{n^5}}{3 + \dfrac{4}{n^4} - \dfrac{18}{n^5}} = \frac{\displaystyle\lim_{n\to\infty} \left(4 + \frac{5}{n^3} - \frac{6}{n^5}\right)}{\displaystyle\lim_{n\to\infty} \left(3 + \frac{4}{n^4} - \frac{18}{n^5}\right)} = \frac{4}{3}.$$

In general, for sequences of the form of the previous example it is useful to divide both numerator and denominator by the highest power of n which occurs in the denominator.

Example As another illustration of the idea in the previous example, we have

$$\lim_{n\to\infty} \frac{3n^2 + 2n - 1}{2n^3 - 16n} = \lim_{n\to\infty} \frac{\dfrac{3}{n} + \dfrac{2}{n^2} - \dfrac{1}{n^3}}{2 - \dfrac{16}{n^2}} = \frac{0}{2} = 0.$$

Definition If $\lim_{n\to\infty} a_n$ exists, we say the sequence $\{a_n\}$ *converges*. If the sequence $\{a_n\}$ does not have a limit, we say the sequence *diverges*.

An important class of divergent sequences are those for which a limit does not exist either because the terms grow without an upper bound or because they decrease without any lower bound, as defined in the following definition.

Definition A sequence $\{a_n\}$ is said to *diverge to infinity* if for any real number M there exists an integer N such that $a_n > M$ whenever $n > N$, in which case we write

$$\lim_{n\to\infty} a_n = \infty.$$

A sequence $\{a_n\}$ is said to *diverge to negative infinity* if for any real number M there exists an integer N such that $a_n < M$ whenever $n > N$, in which case we write

$$\lim_{n\to\infty} a_n = -\infty.$$

Example Clearly

$$\lim_{n\to\infty} n^p = \infty$$

for any value of $p > 0$. For given any M, we need only take

$$N = \left\lfloor \sqrt[p]{|M|} \right\rfloor$$

to guarantee that $a_n > M$ whenever $n > N$.

Example We have

$$\lim_{n\to\infty} 2^n = \infty$$

since, given any M, $2^n > M$ for all n if $M \leq 0$ and $2^n > M$ provided

$$n > \frac{\log_{10}(M)}{\log_{10}(2)}$$

if $M > 0$.

Suppose the sequence $\{a_n\}$ diverges and $k \neq 0$ is a constant. Then the sequence $\{ka_n\}$ must also diverge since if $\{ka_n\}$ converged, then the sequence with nth term

$$\frac{1}{k}(ka_n) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges and $k \neq 0$ is a constant, then the sequence $\{ka_n\}$ also diverges.

If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ converges, then the sequence $\{a_n + b_n\}$ also diverges since, if it converged, then the sequence with nth term

$$(a_n + b_n) - b_n = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges. Similarly, the sequence $\{a_n - b_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ converges, then the sequences $\{a_n + b_n\}$ and $\{a_n - b_n\}$ both diverge.

Suppose the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and

$$\lim_{n\to\infty} b_n \neq 0. \tag{1.2.17}$$

Now (1.2.17) implies that we can find an integer N such that $b_n \neq 0$ for all $n > N$. So if the sequence $\{a_n b_n\}$ converged, then the sequence with, for $n > N$, nth term,

$$\frac{1}{b_n}(a_n b_n) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges. Hence $\{a_n b_n\}$ must diverge.

Proposition If the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and

$$\lim_{n\to\infty} b_n \neq 0,$$

then the sequence $\{a_n b_n\}$ diverges

Finally, if the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and $b_n \neq 0$ for all n, then the sequence

$$\left\{\frac{a_n}{b_n}\right\}$$

diverges since, if it converged, the sequence with nth term

$$b_n\left(\frac{a_n}{b_n}\right) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and $b_n \neq 0$ for all n, then the sequence

$$\left\{\frac{a_n}{b_n}\right\}$$

diverges.

Example Consider

$$\lim_{n\to\infty} \frac{4n^3+n-2}{5n^2-7n} = \lim_{n\to\infty} \frac{4n+\dfrac{1}{n}-\dfrac{2}{n^2}}{5-\dfrac{7}{n}}. \tag{1.2.18}$$

Now

$$\lim_{n\to\infty} 4n = \infty$$

and

$$\lim_{n\to\infty} \left(\frac{1}{n}-\frac{2}{n^2}\right) = 0,$$

so

$$\lim_{n\to\infty} \left(4n+\frac{1}{n}-\frac{2}{n^2}\right) = \infty.$$

Moreover,

$$\lim_{n\to\infty} \left(5-\frac{7}{n}\right) = 5.$$

Thus the numerator in (1.2.18) diverges while the denominator converges. Hence the ratio diverges. In fact, it should be clear that

$$\lim_{n\to\infty} \frac{4n^3+n-2}{5n^2-7n} = \lim_{n\to\infty} \frac{4n+\dfrac{1}{n}-\dfrac{2}{n^2}}{5-\dfrac{7}{n}} = \infty.$$

Note that in the previous example it was once again useful to divide numerator and denominator by the highest power of n in the denominator.

Example We have

$$\lim_{n\to\infty} \frac{15-26n^5}{13+n^2} = \lim_{n\to\infty} \frac{\dfrac{15}{n^2}-26n^3}{\dfrac{13}{n^2}+1} = -\infty.$$

Example The absolute values of the terms of the sequence $\{(-2)^n\}$ grow without bound, and so the sequence diverges. However, since the terms alternate in sign, the sequence neither diverges to ∞ nor to $-\infty$.

Monotone sequences

It is sometimes possible to determine that a given sequence converges without explicitly computing the limit. One important case involves *monotone sequences*.

Definition We say a sequence $\{a_n\}$ is *monotone increasing* if $a_n \leq a_{n+1}$ for all n. We say a sequence $\{a_n\}$ is *monotone decreasing* if $a_n \leq a_{n+1}$ for all n. We say a sequence is *monotone* if it is either monotone increasing or monotone decreasing.

Now suppose $\{a_n\}$ is a monotone increasing sequence. For such a sequence there either exists a number P such that $a_n \leq P$ for all n or there does not exist such an P. In the latter case, given any real number M, it is then possible to find integer N such that $a_N > M$. Since the sequence is monotone, it follows that $a_n > M$ for all $n > N$, and so the sequence diverges to infinity. On the other hand, if there does exist a number P such that $a_n \leq P$ for all n, then there in fact exists a number B such that $a_n \leq B$ for all n and $B \leq P$ for any number P with the property that $a_n \leq P$ for all n. The existence of B, known as the *least upper bound* of the sequence $\{a_n\}$, is not at all obvious; indeed, the subtle properties of the real numbers that imply the existence of B were not fully understood until the middle part of the 19th century. However, given the existence of B, it is easy to see that given any $\epsilon > 0$, there exists a integer N for which $a_N > B - \epsilon$ (if not, then $B - \epsilon$ would be an upper bound for the sequence smaller than B). Since the sequence is monotone increasing and $a_n < B$ for all n, it follows that

$$|a_n - B| < \epsilon$$

for all $n > N$. That is, we have shown that the sequence converges and

$$\lim_{n\to\infty} a_n = B.$$

Similar results hold for sequences which are monotone decreasing.

Monotone sequence theorem Suppose the sequence $\{a_n\}$ is monotone. If the sequence is monotone increasing and there exists a number P such that $a_n \leq P$ for all n, then the sequence converges. If the sequence is monotone increasing and no such number P exists, then

$$\lim_{n\to\infty} a_n = \infty.$$

If the sequence is monotone decreasing and there exists a number Q such that $a_n \geq Q$ for all n, then the sequence converges. If the sequence is monotone decreasing and no such number Q exists, then

$$\lim_{n\to\infty} a_n = -\infty.$$

Example As we shall see in Sections 1.4 and 1.5, we often work with sequences without having an explicit formula for each term in the sequence. For example, suppose all we know about the sequence $\{a_n\}$ is that $a_1 = 4$ and

$$a_{n+1} = \frac{1}{2}a_n$$

for $n = 1, 2, 3, \ldots$. That is, the first term in the sequence is 4 and then each successive term is one-half of its predecessor. Thus

$$\begin{aligned} a_1 &= 4, \\ a_2 &= 2, \\ a_3 &= 1, \\ a_4 &= \frac{1}{2}, \end{aligned}$$

and so on. Hence $\{a_n\}$ is monotone decreasing. Moreover, every term in the sequence is positive, so $a_n \geq 0$ for all n. Thus, by the Monotone Sequence Theorem, $\{a_n\}$ converges. Moreover, note that

$$a_{n+1} = \frac{1}{2} a_n$$

implies that

$$\lim_{n \to \infty} a_{n+1} = \frac{1}{2} \lim_{n \to \infty} a_n. \tag{1.2.19}$$

If we let

$$L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1},$$

then (1.2.19) becomes

$$L = \frac{1}{2} L.$$

Hence $L = 0$. That is,

$$\lim_{n \to \infty} a_n = 0.$$

Problems

1. For each of the following, find a general expression for the nth term of a sequence which would yield these values as the first four terms.

   (a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

   (b) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   (c) $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$

   (d) $-\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, \ldots$

2. For each of the following, decide whether the given sequence converges or diverges. If the sequence converges, find its limit.

   (a) $a_n = \dfrac{1}{3^n}$, $n = 0, 1, 2, \ldots$

   (b) $a_n = \pi^n$, $n = 0, 1, 2, \ldots$

   (c) $b_n = \dfrac{3n - 1}{2n + 6}$, $n = 1, 2, 3, \ldots$

   (d) $c_n = \cos(\pi n)$, $n = 0, 1, 2, \ldots$

   (e) $a_n = \dfrac{3n^4 - 6n^3 + 1}{5n^3 + n^2 + 2}$, $n = 1, 2, 3, \ldots$

   (f) $b_n = \dfrac{2n^5 - 3n^2 + 23}{7n^5 + 13n^4 - 12}$, $n = 1, 2, 3, \ldots$

(g) $c_n = \dfrac{45 - 16n^2}{13 + 5n + 6n^3}$, $n = 1, 2, 3, \ldots$

(h) $b_n = \dfrac{3n + 1}{\sqrt{4n^2 + 1}}$, $n = 1, 2, 3, \ldots$

(i) $a_n = (-2)^{2n+1}$, $n = 1, 2, 3, \ldots$

(j) $a_n = \dfrac{10 - 16n^3}{1 + n^2}$, $n = 1, 2, 3, \ldots$

(k) $a_n = \sqrt{\dfrac{3n^2 + n - 6}{5n^2 + 16}}$, $n = 1, 2, 3, \ldots$

(l) $b_n = \dfrac{(-1)^n}{5^n}$, $n = 0, 1, 2, \ldots$

3. Explain why

$$-1 \le \frac{\sin(n)}{n} \le 1$$

for $n = 1, 2, 3, \ldots$. What can you conclude about $\displaystyle\lim_{n\to\infty} \frac{\sin(n)}{n}$?

4. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$, $n = 1, 2, 3, \ldots$.

   (a) Compute a_1, a_2, a_3, a_4, and a_5 using a calculator.
   (b) Compute values of a_n for $n = 1, 2, 3, \ldots, 200$.
   (c) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 200$, along with the horizontal line $y = e$.
   (d) Does it seem reasonable that $\displaystyle\lim_{n\to\infty} a_n = e$?
   (e) What is the smallest value of n for which $a_n > e$?
   (f) What is the first value of n for which $|a_n - e| < 0.01$? Recall that $e = 2.71828$ to five decimal places.

5. Let $a_n = n \sin\left(\dfrac{1}{n}\right)$, $n = 1, 2, 3, \ldots$.

   (a) Compute a_1, a_2, a_3, a_4, and a_5 using a calculator.
   (b) Compute values of a_n for $n = 1, 2, 3, \ldots, 200$.
   (c) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 200$, along with the horizontal line $y = 1$.
   (d) Does it seem reasonable that $\displaystyle\lim_{n\to\infty} a_n = 1$?
   (e) What is the smallest value of n for which $a_n > 0.999$?
   (f) What is the first value of n for which $|a_n - 1| < 0.0001$?

6. Let $a_n = 1.01^n$ and $b_n = 0.99^n$ for $n = 0, 1, 2, \ldots$. On the same graph, plot the points (n, a_n) and (n, b_n) for $n = 0, 1, 2, \ldots, 200$. How do these two plots compare? Do the sequences converge?

7. Let $a_n = \dfrac{10^n}{n!}$ for $n = 1, 2, 3, \ldots$.

   (a) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 100$.
   (b) From the picture in part (a), can you guess $\displaystyle\lim_{n\to\infty} a_n$?
   (c) What is the maximum value of a_n for $n = 1, 2, 3, \ldots, 100$?

(d) Can you see why

$$\lim_{n\to\infty} \frac{k^n}{n!} = 0$$

for any constant k?

8. Consider the sequence $\{a_n\}$ with $a_1 = 10$ and

$$a_{n+1} = \frac{1}{3}a_n$$

for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots 50$. Do you think this sequence has a limit? Can you verify this?

9. Consider the sequence $\{a_n\}$ with $a_1 = 2$ and

$$a_{n+1} = 2a_n$$

for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 50$. Can you find the limit of this sequence using the same method you used in part Problem 8? Does this sequence have a limit?

10. Consider the sequence $\{a_n\}$ with $a_1 = 0.9$ and

$$a_{n+1} = 2a_n(1 - a_n)$$

for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 100$. Do you think this sequence has a limit? If so, can you find it?

11. In each of the following, for an arbitrary $\epsilon > 0$, find the smallest integer N for which $|a_n - L| < \epsilon$ whenever $n > N$. Verify that your value for N works in the particular case $\epsilon = 0.001$.

(a) $a_n = 1 - \dfrac{1}{n}$, $L = 1$

(b) $a_n = 0.98^n$, $L = 0$

(c) $a_n = \dfrac{1}{n^2}$, $L = 0$

(d) $a_n = \dfrac{3n^3 - 1}{n^3}$, $L = 3$

12. Show that for any $-1 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$.

13. Find sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ and $\{b_n\}$ both diverge, but $\{a_n + b_n\}$ converges.

14. Find sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ diverges, $\{b_n\}$ converges, and $\{a_n b_n\}$ converges.