Difference Equations to Differential Equations

Section 2.2

Trigonometric Functions

Many processes in nature are cyclic. A pendulum oscillates back and forth, repeating its motion over and over; a weight hanging at the end of a spring bobs up and down; the Earth repeats its orbit about the Sun every 365 days; a population of arctic wolves has periods of growth followed by periods of decrease, following the fluctuations in the population of their prey; the monthly rainfall at an agricultural research station varies cyclically over the years and over the decades. To model such natural behavior, a mathematician needs functions which repeat their values over intervals of fixed length. These functions are the periodic functions. Precisely, a function f is *periodic* if there is a fixed constant T such that $f(t+T) = f(t)$ for every value of t in the domain of f. The smallest such positive T for which this property holds is called the *period* of f.

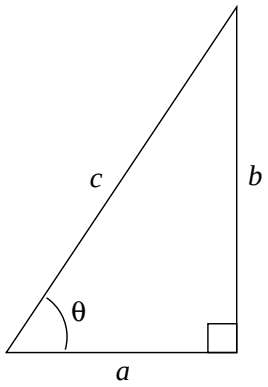

Figure 2.2.1 A right triangle

The class of periodic functions that we will consider in this section are the trigonometric functions. Although these functions were originally invented to work with problems of measurement, their importance in modern mathematics stems more from their periodic behavior. We will begin with a definition in terms of measuring the sides of a right triangle. Consider a right triangle with legs of lengths a and b and hypotenuse of length c. Moreover, suppose, as in Figure 2.2.1, the angle opposite the leg of length b has measure θ. Then we define the *sine* of θ, which we write as $\sin(\theta)$, by

$$\sin(\theta) = \frac{b}{c} \tag{2.2.1}$$

and the *cosine* of θ, which we write as $\cos(\theta)$, by

$$\cos(\theta) = \frac{a}{c}. \tag{2.2.2}$$

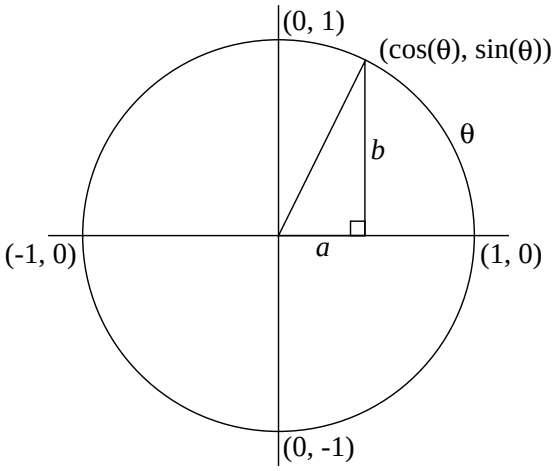

Figure 2.2.2 A right triangle with a vertex on the unit circle

The properties of similar triangles, known even by the ancient Egyptians and Babylonians, show that these ratios depend only on the value of θ, not on the size of the particular right triangle being measured. Hence if we know the value of θ and the length of just one side of the triangle, and we have access to a table of values for the sine and cosine functions, then it is possible to compute the lengths of the other two sides of the triangle. The ancient Greek mathematicians exploited these facts in order to compute distances which are inaccessible to direct measurement, such as the distance from the earth to the moon and from the earth to the sun.

Since the values of the sine and cosine functions do not depend on the size of any particular right triangle, for the purpose of definitions we may restrict our attention to right triangles with hypotenuses of length one. In particular, if we have a right triangle with legs of lengths a and b and hypotenuse of length 1 (so that $a^2 + b^2 = 1$), then we may draw it in the Cartesian plane with one leg running from $(0, 0)$ to $(a, 0)$ and the other from $(a, 0)$ to (a, b). If θ is the measure of the angle opposite the side of length b, then we have

$$\sin(\theta) = b \tag{2.2.3}$$

and

$$\cos(\theta) = a. \tag{2.2.4}$$

In that case, the vertex (a, b) lies on the unit circle $x^2+y^2 = 1$. In particular, $(\cos(\theta), \sin(\theta)$ is a point on the unit circle centered at the origin. This also gives us a method for measuring angles. We will say that the measure of the angle opposite the side of length b is θ *radians* if the length of the arc of the unit circle from $(a, 0)$ to (a, b) is θ. See Figure 2.2.2.

So far our definitions of sine and cosine include only angles that are between 0 and $\frac{\pi}{2}$ radians. However, the considerations of the previous paragraph show us how to generalize our definitions. Let t be any real number and let C be the unit circle centered at $(0, 0)$. If $t \geq 0$, let (a, b) be the point reached by traversing C a distance of t units in the counterclockwise direction starting from $(1, 0)$. If $t < 0$, let (a, b) be the point reached by traversing C a distance of $|t|$ units in the clockwise direction starting from $(1, 0)$. Note

that if $t \geq 2\pi$ or $t \leq -2\pi$, then we will have to travel around C one or more times. We now define the sine and cosine of t by

$$\sin(t) = b \tag{2.2.5}$$

and

$$\cos(t) = a. \tag{2.2.6}$$

In this way we have sine and cosine defined as functions on the entire real line. That is, both sine and cosine now have domain $(-\infty, \infty)$.

Our final definitions of the sine and cosine functions have several immediate consequences. Most importantly, since the circumference of the unit circle is 2π, both functions are periodic with period 2π. Hence

$$\sin(t + 2\pi) = \sin(t) \tag{2.2.7}$$

and

$$\cos(t + 2\pi) = \cos(t) \tag{2.2.8}$$

for any value of t. Also, since $(\cos(t), \sin(t))$ is a point on the unit circle, we have

$$\sin^2(t) + \cos^2(t) = 1 \tag{2.2.9}$$

for all values of t. Recall that, in this notation,

$$\sin^2(t) = (\sin(t))^2$$

and

$$\cos^2(t) = (\cos(t))^2.$$

We will consider other interesting and useful identities involving sine and cosine in the problems at the end of this section and later on as the need for them arises.

Although numerical approximations of $\sin(t)$ and $\cos(t)$ are easily available from a calculator for any value of t, it is useful to know some exact numerical values for these functions. First of all, directly from the definition we have

$$\sin(0) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1 \qquad \sin(\pi) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1 \quad \sin(2\pi) = 0$$

and

$$\cos(0) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = 0 \qquad \cos(\pi) = -1 \qquad \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \cos(2\pi) = 1.$$

Second, with a little more work, it can be shown that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

and

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

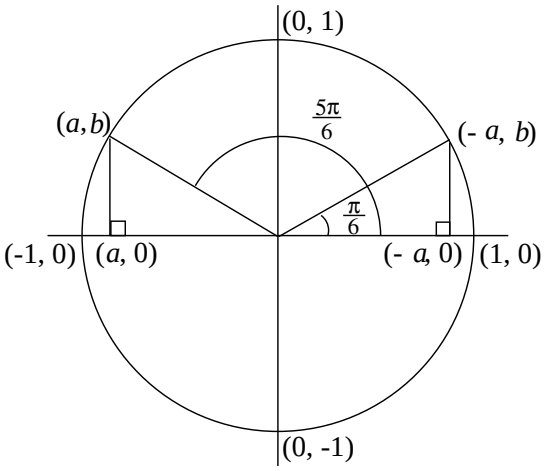

Figure 2.2.3 Finding $\sin\left(\frac{5\pi}{6}\right)$ and $\cos\left(\frac{5\pi}{6}\right)$

Moreover, combining these values with basic knowledge of the geometry of the unit circle, it is possible to find exact numerical values for the values of $\sin(t)$ and $\cos(t)$ for $t = \frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{6}$.

Example Let (a, b) be the point on the unit circle corresponding to the angle $\frac{5\pi}{6}$. Since (a, b) is a distance $\frac{\pi}{6}$ along the unit circle before $(-1, 0)$, the point $(-a, b)$ is a distance $\frac{\pi}{6}$ along the unit circle after $(1, 0)$. Hence

$$-a = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

and

$$b = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Thus

$$\cos\left(\frac{5\pi}{6}\right) = a = -\frac{\sqrt{3}}{2}$$

and

$$\sin\left(\frac{5\pi}{6}\right) = b = \frac{1}{2}.$$

This is all best seen using a picture such as Figure 2.2.3. Note that the triangle with vertices at $(0, 0)$, (a, b), and $(a, 0)$ is congruent to the triangle with vertices at $(0, 0)$, $(-a, b)$, and $(-a, 0)$.

Of course, because both sine and cosine have period 2π, it is also easy to find exact values for $\sin(t)$ and $\cos(t)$ if t differs from one of the above values by a multiple of 2π.

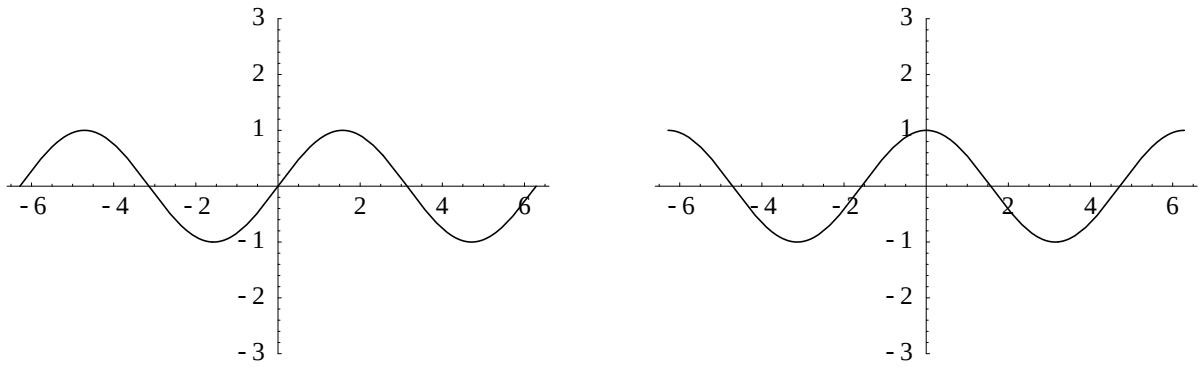

Figure 2.2.4 Graphs of $y = \sin(t)$ and $y = \cos(t)$

Graphs of sine and cosine

The graphs of $y = \sin(t)$ and $y = \cos(t)$ are shown in Figures 2.2.4. Since both functions have period 2π, the graphs will continue this behavior as t goes to $-\infty$ or ∞, completing one oscillation over every interval of length 2π.

The only difference between the graph of $y = a\sin(t)$, where $a > 0$, and the graph of $y = \sin(t)$ is that the former oscillates between $-a$ and a instead of between -1 to 1. In general, for any constant $a \neq 0$, the graph of $y = a\sin(t)$ oscillates between $-|a|$ and $|a|$. We call $|a|$ the *amplitude* of the function $y = a\sin(t)$. Of course, if $a < 0$, then the graph of $y = a\sin(t)$ is the graph of $y = |a|\sin(t)$ reflected about the t-axis.

Example The graph of $y = 2\sin(t)$ is shown in Figure 2.2.5.

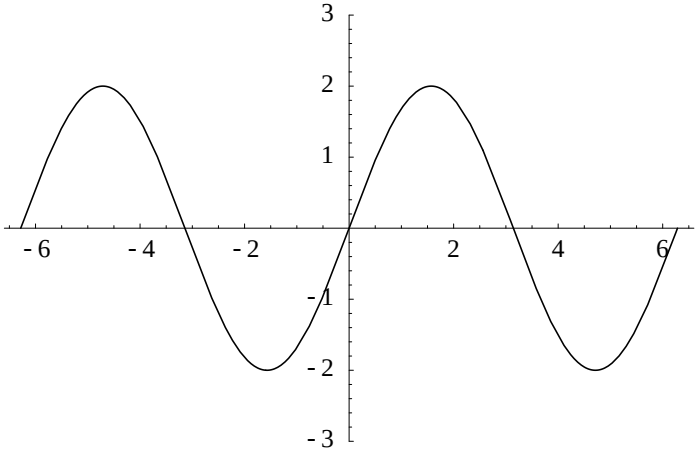

Figure 2.2.5 Graph of $y = 2\sin(t)$

Now consider the graph of the function $y = \sin(bt)$. Since the sine function has period 2π, this function goes through one complete oscillation as t goes from 0 to $\frac{2\pi}{|b|}$. That is, $y = \sin(bt)$ has period $\frac{2\pi}{|b|}$. Hence, if $b > 0$ the only difference between the graphs of $y = \sin(t)$ and $y = \sin(bt)$ is the length of the period of oscillation. If $b < 0$, we may use

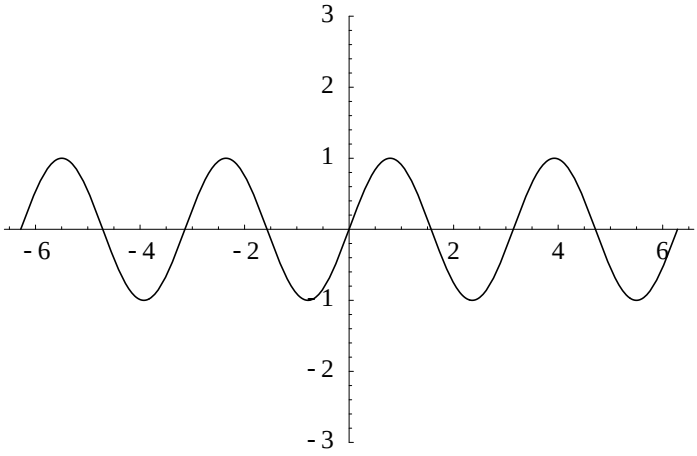

Figure 2.2.6 Graph of $y = \sin(2t)$

the fact that

$$\sin(bt) = \sin(-|b|t) = -\sin(|b|t),$$

which follows from Problem 10.

Example The graph of $y = \sin(2t)$ is shown in Figure 2.2.6.

Finally, consider the graph of $y = \sin(t - c)$. As mentioned in Section 2.1, the effect of the c is to shift the graph $y = \sin(t)$ horizontally by $|c|$ units, to the right if $c > 0$ and to the left if $c < 0$. We call c the *phase angle*.

Example The graph of $y = \sin(t - \pi)$ is shown in Figure 2.2.7.

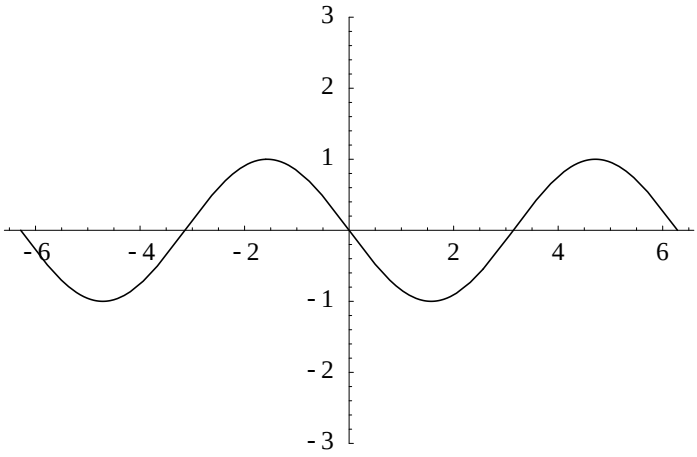

Figure 2.2.7 Graph of $y = \sin(t - \pi)$

Summarizing the previous comments, the function $y = a\sin(b(t - c))$ has amplitude $|a|$, period $\frac{2\pi}{|b|}$, and phase angle c.

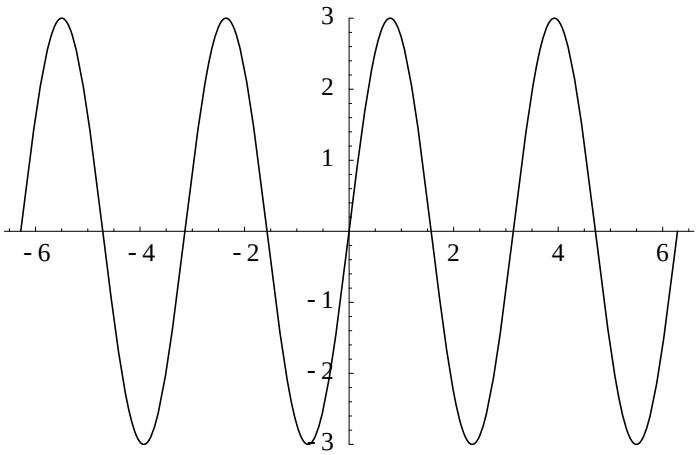

Figure 2.2.8 Graph of $y = -3\sin(2t - \pi)$

Example Consider the function $f(t) = -3\sin(2t - \pi)$. If we write $f(t)$ in the form

$$f(t) = -3\sin\left(2\left(t - \frac{\pi}{2}\right)\right),$$

then we see that f has amplitude 3, period π, and phase angle $\frac{\pi}{2}$. The graph of f is shown in Figure 2.2.8.

Similar remarks hold for the graph of the function $y = a\cos(b(t-c))$, the only difference being that, since $\cos(-t) = \cos(t)$ for all t (see Problem 10), we have

$$\cos(bt) = \cos(|b|t)$$

even when $b < 0$.

Related functions

The other four trigonometric functions are defined in terms of the sine and cosine functions. The *tangent* function is defined by

$$\tan(t) = \frac{\sin(t)}{\cos(t)}. \tag{2.2.10}$$

Note that $\tan(t)$ is the slope of the line from $(0, 0)$ to $(\cos(t), \sin(t))$. The graph of $y = \tan(t)$ has vertical asymptotes at every value of t for which $\cos(t) = 0$, as can be seen in Figure 2.2.9.

The *cotangent* function is the reciprocal of the tangent function; namely,

$$\cot(t) = \frac{1}{\tan(t)} = \frac{\cos(t)}{\sin(t)}. \tag{2.2.11}$$

Finally, the *secant* and *cosecant* functions are the reciprocals of the cosine and sine functions, respectively. Hence

$$\sec(t) = \frac{1}{\cos(t)} \tag{2.2.12}$$

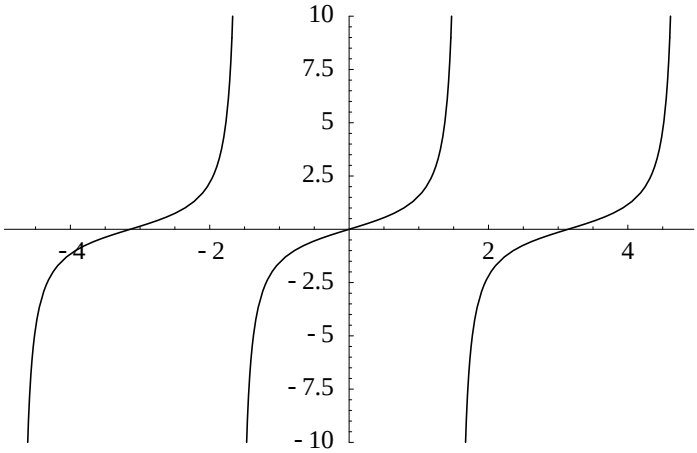

Figure 2.2.9 Graph of $y = \tan(t)$

and

$$\csc(t) = \frac{1}{\sin(t)}. \tag{2.2.13}$$

As with the tangent function, the graph of $y = \sec(t)$ has vertical asymptotes at all points t where $\cos(t) = 0$, as seen in Figure 2.2.10.

Clearly both the secant and cosecant functions have period 2π. However, the tangent and cotangent functions both have period π. We will leave that fact, along with the graphs of the cotangent and cosecant functions, to the problems at the end of this section.

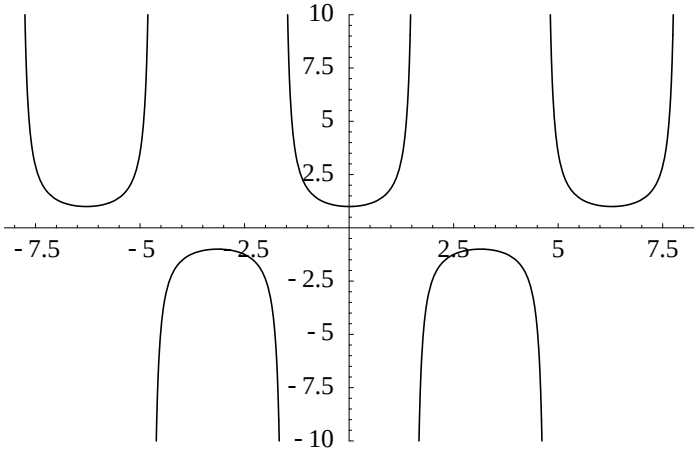

Figure 2.2.10 Graph of $y = \sec(t)$

Periodic motion

As mentioned earlier, many natural phenomena change in a periodic fashion. For example, suppose we have a pendulum and for a given time t we let $x(t)$ represent the angle between the current position of the pendulum and its rest position, taking x to be positive if the

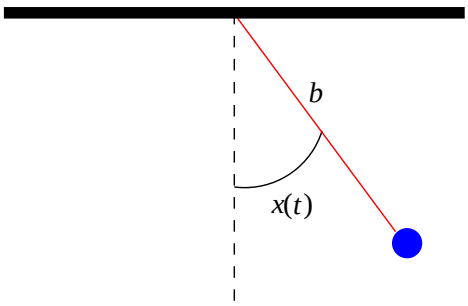

Figure 2.2.11 A pendulum

pendulum is to the right of its rest position and negative otherwise (see Figure 2.2.11). If initially the pendulum is held at a small angle $\alpha > 0$ and then released, that is, $x(0) = \alpha$, then, if we ignore friction, it can be shown that

$$x(t) = \alpha \cos\left(\sqrt{\frac{g}{b}}t\right) \tag{2.2.14}$$

where g is the acceleration due to gravity (32 feet per second per second or 9 .8 meters per second per second) and b is the length of the pendulum. Actually, this is an approximation which holds very well for small values of α. Note that the period of x, namely,

$$\frac{2\pi}{\sqrt{\frac{g}{b}}} = 2\pi\sqrt{\frac{b}{g}},$$

does not depend upon the amplitude α. This is an important fact, supposedly first noticed by Galileo, which is crucial in the operation of pendulum clocks. We will consider this problem more closely in Chapter 8, where we will derive (2.2.14) and see exactly how the approximation enters the picture.

Periodic motions need not always be as simple as the motion of a pendulum. Consider, for example, the motion of a molecule of air as a sound wave passes. The action of the sound wave causes a particular molecule of air to oscillate back and forth about some equilibrium position. If we let $x(t)$ represent the position of the air molecule at time t, with $x = 0$ corresponding to the equilibrium position and x considered to be positive in one direction from the equilibrium position and negative in the other, then for many sounds x will be a periodic function of t. In general, this will be true for musical sounds, but not true for sounds we would normally classify as noise. Moreover, even if x is a periodic function, it need not be simply a sine or cosine function. The graph of x for a musical sound, although periodic, may be very complicated. However, many simple sounds, such as the sound of a tuning fork, are represented by sine curves. For example, if x is the displacement of an air molecule for a tuning fork which vibrates at 440 cycles per second with a maximum displacement from equilibrium of 0.002 centimeters, then

$$x(t) = 0.002 \sin(880\pi t).$$

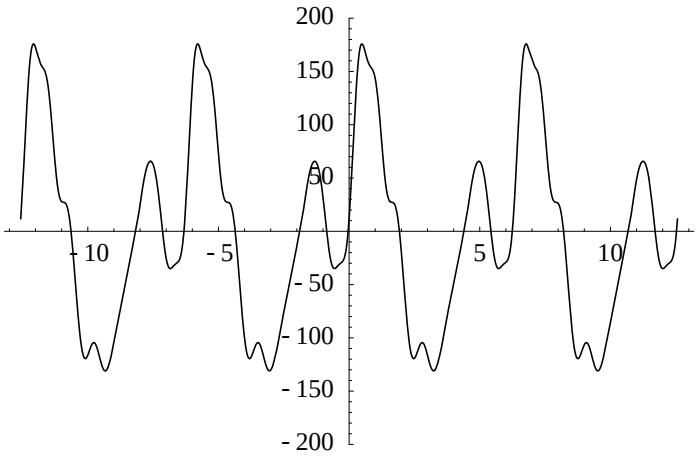

Figure 2.2.12 Graph of the air displacement due to the organ note C_3

Notice that this function has period $\frac{2\pi}{880\pi} = \frac{1}{440}$, and hence has a frequency of 440 cycles per second.

In the early part of the 19th century, Joseph Fourier (1768-1830) showed that the story does not end here. Fourier demonstrated that any "nice" periodic curve (for example, one which is connected) can be approximated as closely as desired by a sum of sine and cosine functions. In particular, this means that for any musical sound the function x may be approximated well by a sum of sine and cosine functions. For example, in his book *The Science of Musical Sounds* (Macmillan, New York, 1926), Dayton Miller shows that, with an appropriate choice of units,

$$\begin{aligned} x(t) = {} & 22.4\sin(t) + 94.1\cos(t) + 49.8\sin(2t) - 43.6\cos(2t) + 33.7\sin(3t) \\ & - 14.2\cos(3t) + 19.0\sin(4t) - 1.9\cos(4t) + 8.9\sin(5t) - 5.22\cos(5t) \\ & - 8.18\sin(6t) - 1.77\cos(6t) + 6.40\sin(7t) - 0.54\cos(7t) + 3.11\sin(8t) \\ & - 8.34\cos(8t) - 1.28\sin(9t) - 4.10\cos(9t) - 0.71\sin(10t) - 2.17\cos(10t) \end{aligned}$$

gives a very good approximation to the displacement curve of a sound wave generated by the tone C_3 of an organ pipe. From the graph of x, shown in Figure 2.2.12, we can see its complexity as well as its periodicity. Notice that the terms in this expression for $x(t)$ are written in pairs with frequencies which are always integer multiples of the frequency of the first pair. This is a general fact which is part of Fourier's theory; if we added more terms to obtain more accuracy, the next terms would be of the form $a\sin(11t) + b\cos(11t)$ for some constants a and b. Notice also that the amplitudes of the sine and cosine curves tend to decrease as the frequencies are increasing. As a consequence, the higher frequencies have less impact on the total curve. Put another way, Fourier's theorem says that every musical sound is the sum of simple tones which could be generated by tuning forks. Hence in theory, although certainly not in practice, the instruments of any orchestra could all be replaced by tuning forks. On a more practical level, Fourier's analysis of periodic functions has been fundamental for the development of such modern conveniences as radios, televisions, stereos, and compact disc players. Unfortunately, this is a story which will have to be told elsewhere.

Problems

1. Find the exact values of $\sin(t)$, $\cos(t)$, $\tan(t)$, and $\sec(t)$ for the following values of t.

   (a) $\dfrac{4\pi}{3}$ (b) $\dfrac{7\pi}{6}$

   (c) $\dfrac{2\pi}{3}$ (d) $-\dfrac{\pi}{4}$

   (e) $-\dfrac{2\pi}{3}$ (f) $\dfrac{21\pi}{4}$

   (g) $\dfrac{11\pi}{6}$ (h) $-\dfrac{11\pi}{6}$

2. Sketch the graph of each of the following functions over an interval that contains at least one period of the function both to the right and to the left of the vertical axis. Also, identify the amplitude, period, and phase angle of each curve.

   (a) $y = \sin(3t)$ (b) $y = 3\cos(2t)$

   (c) $y = \cos(t - \pi)$ (d) $x = \sin(2t) + 1$

   (e) $x = 4\sin(\pi t)$ (f) $y = -2\cos(2t - \pi)$

   (g) $x = 5\sin(2t + \pi)$ (h) $y = -3\sin(2\pi t)$

3. Starting with the identity $\sin^2(x) + \cos^2(x) = 1$, explain why

$$1 + \tan^2(x) = \sec^2(x).$$

4. The addition formulas for sine and cosine are

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

   and

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

   Use these to derive the double-angle formulas:

   (a) $\sin(2x) = 2\sin(x)\cos(x)$ (b) $\cos(2x) = \cos^2(x) - \sin^2(x)$

5. Use the double-angle formulas of Problem 4 to derive the half-angle formulas:

   (a) $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$ (b) $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

6. Use the addition formulas of Problem 4 to derive the shift formulas:

   (a) $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos(x)$ (b) $\cos\left(x - \dfrac{\pi}{2}\right) = \sin(x)$

   (c) $\sin\left(x + \dfrac{\pi}{2}\right) = \cos(x)$ (d) $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin(x)$

7. Can you picture the identities of Problem 6 in terms of the definitions of sine and cosine using the unit circle? What do these identities say about the relationship between the graphs of sine and cosine?

8. Using the addition formulas of Problem 4, show that the tangent and cotangent functions have period π. That is, show that

$$\tan(t+\pi) = \tan(t)$$

and

$$\cot(t+\pi) = \cot(t)$$

for all values of t.

9. Graph each of the following functions.

(a) $y = \tan(2t)$ (b) $y = \cot(t)$

(c) $y = \tan\left(\dfrac{t}{2}\right)$ (d) $y = \csc(t)$

(e) $x = \sec(2t)$ (f) $y = \tan(4t) + 3$

10. Using the definitions of sine and cosine, convince yourself that

$$\sin(-x) = -\sin(x)$$

and

$$\cos(-x) = \cos(x)$$

for all values of x. Now sketch the graphs of $y = \sin(-3x)$ and $y = \cos(\pi - x)$.

11. According to Dayton Miller in *The Science of Musical Sounds*, the function

$$x(t) = 151\sin(t) - 67\cos(t) + 24\sin(2t) + 55\cos(2t) + 27\sin(3t) + 5\cos(3t)$$

gives a good approximation to the shape of the displacement curve for the tone B_4 played on the E string of a violin.

(a) Graph each of the individual terms of x on the interval $[-15, 15]$. Use a common scale for the vertical axis.

(b) Graph x on $[-15, 15]$.

(c) Graph x and its individual terms (a total of 7 graphs) together on the interval $[-15, 15]$.

12. Suppose we define a function f by saying that it is periodic with period 1 and that $f(x) = 1 - 2x$ for $0 \le x < 1$.

(a) Sketch the graph of f over the interval $[-3, 3]$.

(b) Let

$$g_n(x) = 2\left(\frac{1}{\pi}\sin(2\pi x) + \frac{1}{2\pi}\sin(4\pi x) + \frac{1}{3\pi}\sin(6\pi x) + \cdots + \frac{1}{n\pi}\sin(2n\pi x)\right)$$

for $n = 1, 2, 3, \ldots$. For example,

$$g_1(x) = \frac{2}{\pi}\sin(2\pi x),$$

$$g_2(x) = \frac{2}{\pi}\sin(2\pi x) + \frac{1}{\pi}\sin(4\pi x),$$

and

$$g_3(x) = \frac{2}{\pi}\sin(2\pi x) + \frac{1}{\pi}\sin(4\pi x) + \frac{2}{3\pi}\sin(6\pi x).$$

What is the period of g_n? Graph g_1, g_2, g_3, g_4, g_5, and g_{10} over the interval $[-3, 3]$.

(c) What do you think happens to g_n as n gets large?

13. Graph $f(x) = \lfloor \sin(x) \rfloor$ on the interval $[-\pi, \pi]$.

14. For an interesting account of sound waves, Fourier's theorem, and related ideas in electromagnetism, read Chapters 19 ("The Sine of G Major") and 20 ("Mastery of the Ether Waves") in Morris Kline's *Mathematics in Western Culture* (Oxford University Press, 1953).