

A Mathematical Explanation of the Polar Planimeter of Jakob Amsler-Laffon

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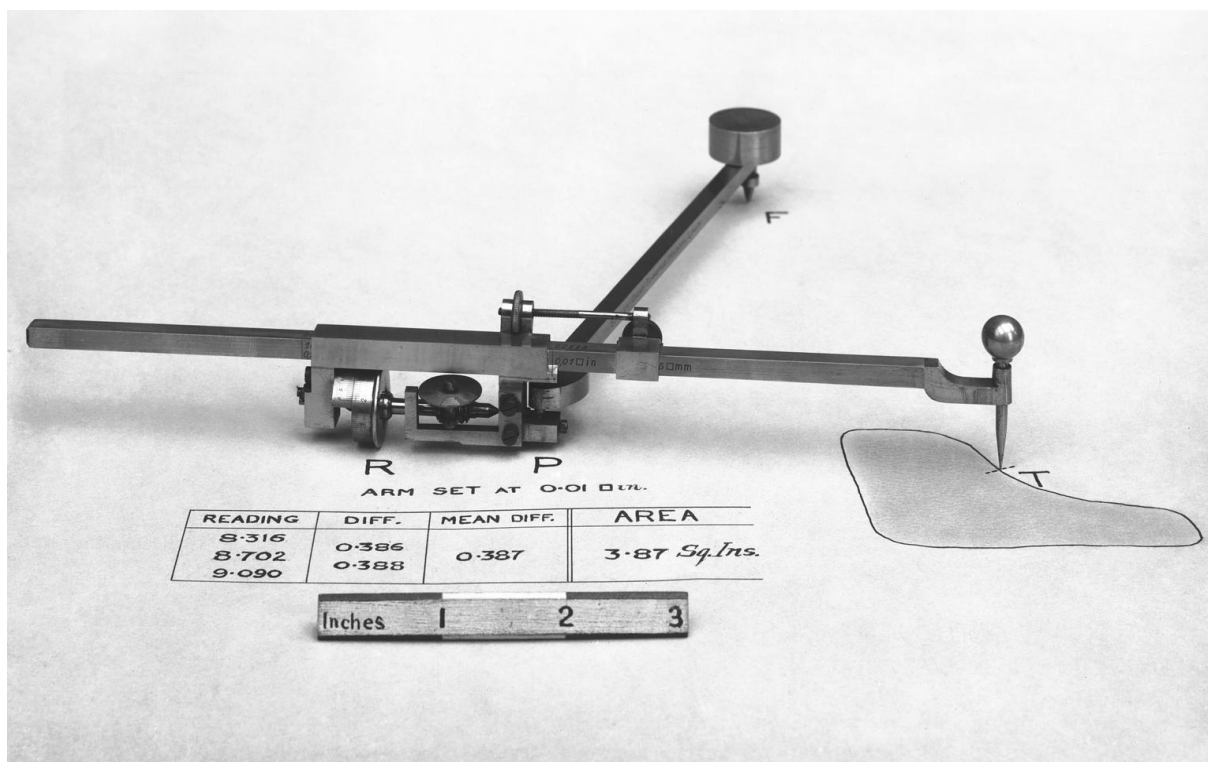
ABSTRACT

The Amsler polar planimeter is a classical mechanical instrument designed to determine the area enclosed by a planar curve through the tracing of its boundary. Invented in the nineteenth century by Jakob Amsler-Laffon, the device constitutes one of the earliest realizations of analog computation. Its operation is rooted in the geometry of constrained motion and in the theory of line integrals. In particular, the planimeter mechanically evaluates an integral that corresponds to the area formula arising from Green's theorem.

The following *descriptive* study examines the mathematical foundations of the Amsler polar planimeter, including its kinematic structure, differential-geometric interpretation, and its relation to some classical integral formulae.

KEYWORDS

Polar planimeter, Amsler, Stokes' and Green's theorem, vector analysis, area measurement.



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1. INTRODUCTION

1.1. Some historical context

In 1854, the Swiss mathematical physicist *Jakob Amsler-Laffon*¹ (1823-1912), see the photograph in Figure 1, constructed a polar planimeter with which the area A_R of an arbitrarily shaped region R in a plane can be measured *exactly* (i.e. not approximated by the area of a finite number of basic shapes like rectangles), simply by tracing the contour C enclosing that region². See references [1] and [2].

Figure 2 shows the Amsler polar planimeter by *Albert Ott*³, about 100 years younger than Amsler's, but a variant operating according to the same principle he discovered.

Amsler produced over 50,000 units across six variants by the late 19th century, driving widespread adoption in surveying, cartography, and mechanical engineering by the 1860s. See [21].

Various variants of Amsler's polar planimeter can be found on the website [22]. Also, David Green's website [8] provides a detailed overview of various types of planimeters; the site includes specific information on Amsler planimeters.

Figure 1

Jakob Amsler(-Laffon). See biography [10].



It is remarkable that, almost simultaneously with and independently of Amsler, another scientist, *Albert Miller von Hauenfels*, professor at the Leoben Mining Academy in Graz, Austria, devised a similar planimeter, patent granted 13 November 1855, so, somewhat later than Amsler. See [20]. The instrument was manufactured in Vienna by the company *Starke*, makers of surveying and astronomical instruments. Hence, the device became known as the *Miller-Starke planimeter*. One can find a photo of a *Starke & Kammerer* planimeter on the website [18].

Detlef Zerfowski and *John Vossepoel* wrote an important study of the history of planimeters [17]. The authors examine more than thirty lesser-known area-measuring instruments developed

¹ Amsler-Laffon studied first theology and later mathematics and physics at the University of Königsberg, where he was influenced by the physicist Franz Ernst Neumann. Among his fellow students was the later famous physicist Gustav Kirchhoff. He founded a precision instrument company in Schaffhausen and developed scientific and engineering instruments, including testing machines and hydrometric devices. See Wikipedia.

² That the area of a region can be determined merely by tracing its contour was already well known in Amsler's time. In his 1856 paper [2], Amsler refers, among others, to Johannes Oppikofer (1783–1864), who in 1827 constructed a conical planimeter that determines the area by tracing the contour. Amsler notes that this idea is even older, having been discovered in 1814 by the German land surveyor Johann Martin Hermann (1785–1841), but that it subsequently fell into oblivion.

Barbera Haeberlin has created a beautiful digital simulation of the construction and operation of Hermann's conical planimeter, which can be viewed online: <https://www.youtube.com/watch?v=XQXO6edF6sA>. In 2014, an exhibition on Hermann's work was held at the Deutsches Museum in Munich.

Reference [6] primarily describes the history of Hermann and his conical planimeter, but places this history in the broader context of the historical development of various types of planimeters devised and constructed after 1814.

³ Albert Ott was a 19th-century German precision instrument maker based in Kempten who became one of the main industrial producers of planimeters. While the polar planimeter was originally invented by Jakob Amsler-Laffon, Ott's workshop played a key role in manufacturing, refining, and widely distributing Amsler-type planimeters for practical use in engineering and cartography.

between 1814 and 1884. Their work reveals that the evolution of the planimeter was far more diverse than is commonly assumed⁴.



Figure 2

Polar planimeter, type 31, no. 89475, manufactured by A. Ott, Kempten, Bavaria, distributed in the Netherlands by N.V. Wed. J. Ahrend & Zoon. Property of the author.

It is mentioned, the great physicist *James Clerk Maxwell*, see [15, p. 20], who at roughly the same time also designed a planimeter, stared in astonishment at Amsler's polar planimeter. He must undoubtedly have wondered what everyone wonders when seeing a polar planimeter for the first time: how can such a simple instrument measure the area A_R of arbitrarily shaped plane figures *exactly*? Which ingenious mathematical idea is incorporated in this instrument?

Therefore, the main questions we try to answer in this study will be: on the basis of which mathematical principle does an arbitrary closed curve C in the plane determine the area of the enclosed region R and how is this principle applied in Amsler's polar planimeter?

1.2. Explanation of its functioning by Green

However relatively simple the basic construction of a polar planimeter may be, answering these questions requires a rather complex mathematical analysis. In this essay we analyse that remarkable mathematical functioning of Amsler's polar planimeter.

Anyone searching books or the internet for an explanation of its operation will often find the elegant explanation using Green's theorem for the plane, which is also discussed subsequently in this essay. However, we do not know for certain whether Amsler was familiar with this theorem. Green's theorem became widely known in the professional mathematical community only around the same time of Amsler's invention. Although Amsler was a professional mathematician it is

⁴ "A Contribution to the History of Planimeters – Lesser-Known Instruments and Variants" by Detlef Zerfowski and John Vossepel [17] is an important contribution for anyone interested in the early history of planimeters. Published in *Slide Rule Gazette*, Issue 23 (Autumn 2023, pp. 61–102), it offers a comprehensive survey of many little-known area-measuring instruments that are often overlooked in standard historical accounts. Most histories of the planimeter focus on the well-known figures like Amsler and Miller. Zerfowski and Vossepel take a different approach. Rather than concentrating on the famous instruments, they document a wide range of lesser-known devices and variants developed before and alongside the classic planimeters. The article describes more than thirty relatively obscure area-measuring instruments dating from approximately 1814 to 1884.

probable he didn't know yet of its belated republication in the scientific literature⁵.

In any case, he based the operation of his planimeter on a different, more intuitive geometrical idea rather than on the vectorial rotational principle of Stokes that underlies Green's theorem.

1.3. Object of this study

So, in short: the *object of this study* is to investigate the mathematical principles underlying the planimeter:

- (1) To gain insight into the brilliant idea of Amsler, we first discuss his classical description (of which many variants exist), using a combination of *translation* and *rotation* of a line segment, and the conclusion drawn from the surprisingly simple formula with which that rather difficult explanation ends.
- (2) Subsequently shall we present a proof of Amsler's theorem based on concepts from vector analysis, a method that is likely to be more accessible to contemporary historians of mathematical instruments. In the appendix of this study the reader may find the concepts of vector analysis that are used in this proof.

⁵ Green's theorem originated in George Green's 1828 essay *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Green's original 1828 essay was circulated only in a small local edition and remained largely unknown for more than two decades. Its significance became widely recognized only after William Thomson (later Lord Kelvin) rediscovered the work and promoted its republication during the 1850s. The essay was later reprinted in *Crelle's Journal* (*Journal für die reine und angewandte Mathematik*) in three instalments, published in 1850, 1852, and 1854. These publications brought Green's work to the attention of the wider European mathematical community and contributed significantly to the dissemination of Green's theorem and related concepts.

2. CONSTRUCTION OF THE POLAR PLANIMETER

2.1. Amsler's drawings

Figures 3a, 3b, and 3c below show the construction drawings published by Amsler in 1856 in his article [2] on the newly designed polar planimeter.

Figure 4 shows additional details of another drawing by Amsler, in which original Figure 2 is particularly important for understanding the mathematical description that follows below.

Figure 5 shows a simplified instantaneous configuration of the planimeter while tracing contour C of region R, with the most relevant components, but without the measuring wheel W.

Point P is the pole, a fixed reference point during measurement, usually, but not necessarily chosen outside region R.

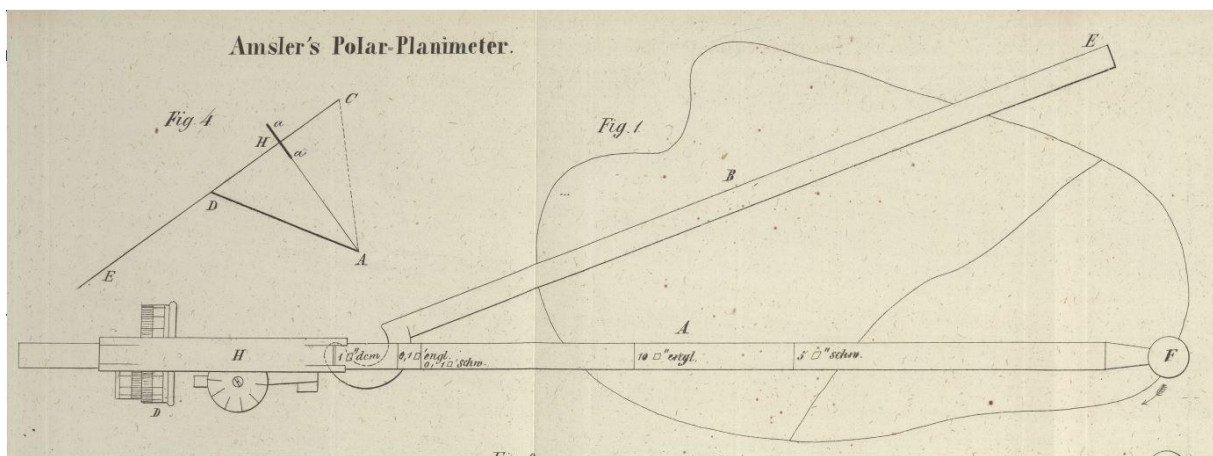
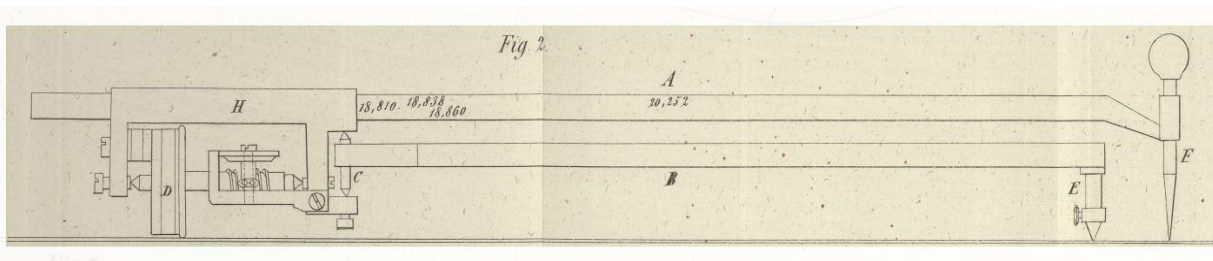


Figure 3a

Amsler's original Figures 1 and 4, showing the tracing arm A, polar arm B, pole E, and tracing point F, as



referenced in [10].

Figure 3b

Amsler's original side view Figure 2 of the polar planimeter, showing the measuring wheel D, transmission to a horizontal dial with vernier scale, and spindle C.

Using Figure 5, one can easily identify the components in Amsler's Figures 3a – 3c. In those Figures, A is the tracing arm ST, B the polar arm PS, C the pivot S, D the measuring wheel W, E the pole P, and F the tracing point T.

The contour C of region R is traced by the tracing point T of a hinged arm system PST, consisting of polar arm PS (red) and tracing arm ST (green), with $ST \leq PS$.

The polar arm rotates around pole P, and the tracing arm rotates around pivot S. The pole P is

preferably chosen such that the angle θ between PS and ST in Figure 5 does not deviate too much from 90° . In any case, θ is kept below 180° , so that the tracing arm cannot align with the polar arm.

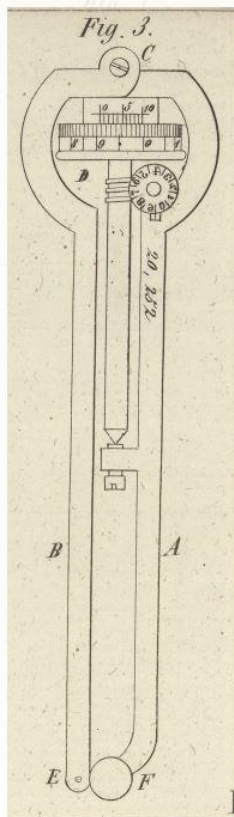


Figure 3c (Left)

Top view (original Figure 3) showing the measuring wheel D and dial displaying total angular displacement.

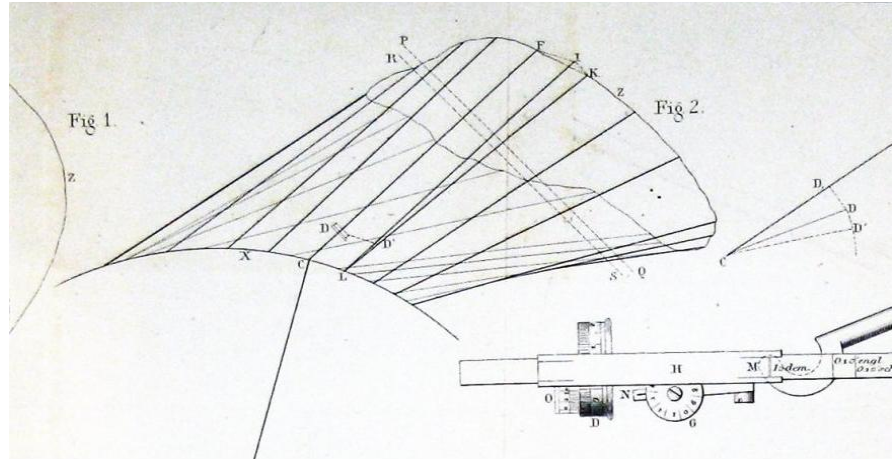


Figure 4

Explanatory detail from Amsler's construction drawing Figure 2. See the rotation at L and perpendicular translation from D to D'. See the American Mathematical Society website [5].

2.2. Explanatory drawings

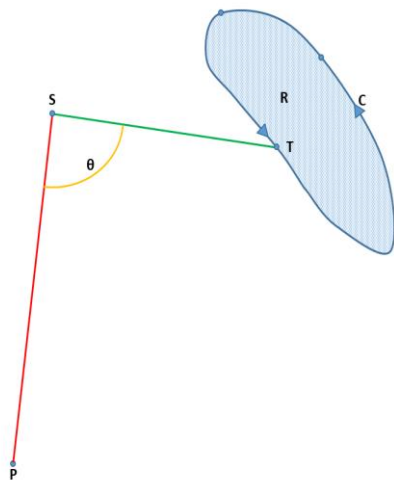


Figure 5 (Left)

Stylized situation. The region R is enclosed by the closed contour C. P is the pole. S is the pivot point. T is the tracing point. PS is the polar arm, and ST is the tracing arm. The tracing point T traverses the contour C once, here in the counterclockwise direction.

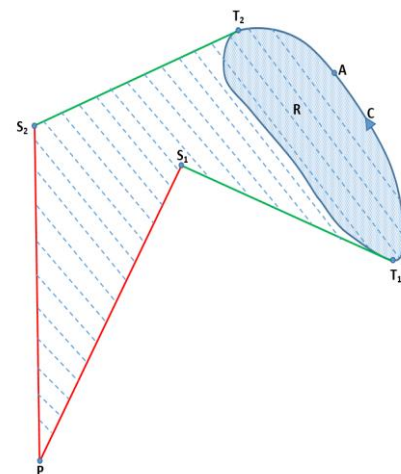


Figure 6 (Right)

The contour C is traversed counterclockwise. As the tracing point T moves from lowest tracing point T_1 to highest tracing point T_2 , via contour point A, the arm system PST, consisting of the polar arm PS and the tracing arm ST, sweeps out the by dotted lines shaded area. A measuring wheel W measures this total area, up to a scale factor. See figure 7. Mind, this total area is larger than the area of region R, we want to measure.

3. THE MEASUREMENT PRINCIPLE OF THE POLAR PLANIMETER

3.1. Tracing the contour

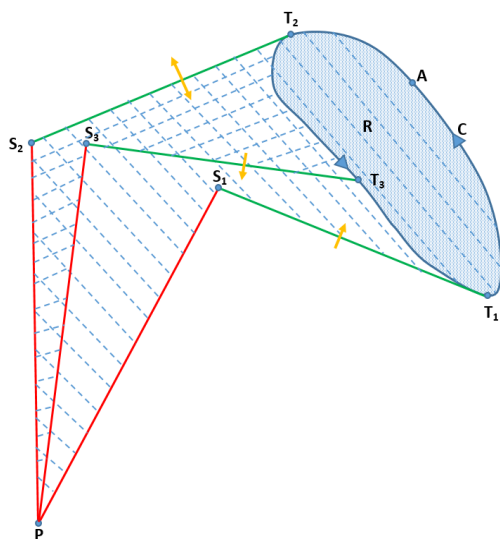
That Amsler's polar planimeter remained an important instrument until around 1980 is shown by the well-known mathematics textbook [3], which was used in the first school year of many Dutch colleges for higher technical education (Hogere Technische School, HTS) during the 1960s and 1970s. For an initial answer to the questions raised in the introduction of this essay, we essentially follow [3, pp. 236–239] from that textbook.

Figure 6 above shows the situation in which the tracing point T follows the contour C . First, T moves along the right-hand side of C , via point A , from lowest tracing point T_1 to highest tracing point T_2 .

The articulated arm system PST then sweeps out the shaded area shown with straight lines in Figure 6, i.e. oblique dotted lines from the lower right to the upper left. This area is clearly considerably larger than the area A_R of the region R to be determined.

This initially too large area is measured by a small measuring wheel W mounted *perpendicular* to the tracing arm ST , indicated in Figure 7 by a small yellow arrow.

When tracing point T traverses the contour C via point A , between the instantaneous tracing points T_1 and T_2 , the measuring wheel W rotates in the forward direction, which corresponds to the usual positive orientation in mathematics and physics. Interestingly, however, Amsler assumes a negative orientation, in which the contour C is traversed in the clockwise direction. For the final result, namely the measuring the area A_R of the region R , this of course makes no essential difference⁶.



3.2. Positive and negative wheel movements

Figure 7

The tracing point T traverses the contour C , via T_3 , from T_2 down to T_1 . When the right-hand side of the contour C was traversed, the measuring wheel W rotated forward, as indicated by the yellow arrow in the tracing arm S_1T_1 . When the tracing arm reaches the position S_2T_2 , the measuring wheel begins to rotate in the reverse direction.

As noted, when the tracing point T follows the right-hand side of the contour C , the measuring wheel W rotates forward. See the yellow arrow in the initial position S_1T_1 of the tracing arm ST in Figure 7.

⁶ In Amsler's polar planimeter, the contour C is traversed in the clockwise direction, contrary to what is customary in mathematics nowadays. For understanding the planimeter, this direction is not essential. However, we shall here adopt the usual mathematically positive direction, i.e. counterclockwise, particularly because the vector-analytical concepts of circulation and curl, which play an underlying role, are based on this orientation. The positive counterclockwise direction corresponds to the positive orientation expressed in the right-hand screw rule or right-hand rule used in electrical engineering.

While traversing the right-hand side of the contour C , the measuring wheel continues to rotate forward, causing W to measure an *increase* in the area swept by the arm system PST.

When the tracing arm ST reaches the highest instantaneous position S_2T_2 , the measuring wheel W begins to rotate backward, which is indicated by a yellow double arrow at the tracing arm S_2T_2 in Figure 7. Then W measures a *decrease* in area.

Figure 7 also shows the situation at the moment when the tracing arm ST reaches the intermediate instantaneous position S_3T_3 . During the traversal of left side of the contour C from T_2 to T_3 , the measuring wheel W rotates backward; the arm system PST has swept out the doubly shaded area, with an extra second set of oblique dotted lines, now from upper right to lower left. This area is subtracted, due to the reverse rotation of the measuring wheel W , from the previously measured, too large area.

When, after tracing point T having completely traversed contour C , the tracing arm ST returns to its initial position S_1T_1 . Then instantaneous point S_3 coincides with S_1 and T_3 coincides with T_1 .

The resulting maximum doubly shaded area of the region $PS_1T_1T_2S_2P$ is subtracted from the total area that was originally counted by W when the tracing point T followed the right-hand side of the contour C (from T_1 via point A to T_2).

So, only the area of the region R to be measured remains as the difference between the two areas. The result is therefore that the measuring wheel W ultimately determines, up to a scale factor, the area A_R of R .

A key question that then arises is: why does the measuring wheel W measure the area A_R at all (up to a scale factor)?

4. MEASURING THE AREA WITH THE WHEEL

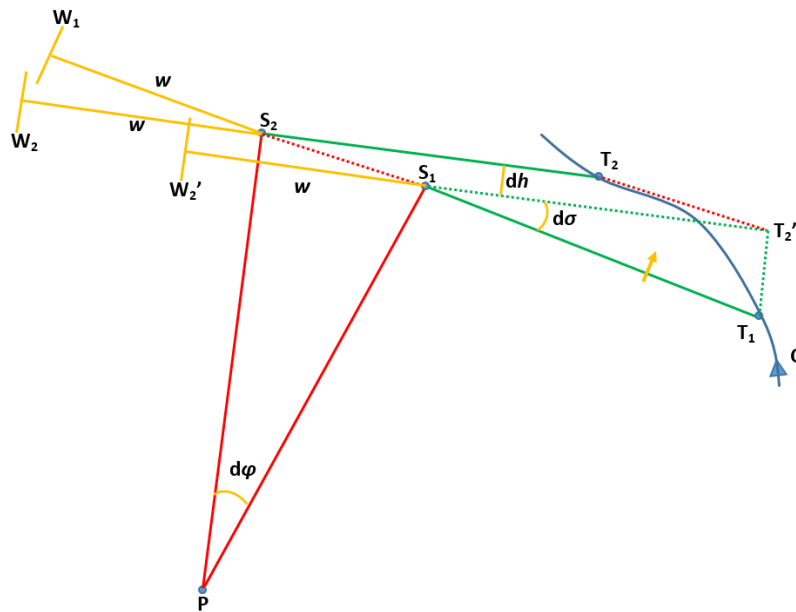


Figure 8

During the tracing of an infinitesimally small segment T_1T_2 of the right-hand side of the contour C , the tracing arm ST performs two motions: a rotation about the pivot point S_1 and a translation over a distance dh perpendicular to S_1T_1 . The articulated system consisting of the polar arm PS and the tracing arm ST sweeps out an area equal to the sum of the areas of the sectors PS_1S_2 and $S_1T_1T_2'$, and of the parallelogram $S_1T_2'T_2S_2$. The small region $T_1T_2'T_2T_1$ outside the contour C that makes (in this case) this sum too large will be ignored in the next calculation. See also in Figure 4, Amsler's original Figure 2 in [2].

4.1. Translation and rotation

We broadly follow the reasoning of reference [3, pp. 236–239].

In Figure 8, the tracing point T moves over an infinitesimal segment of right-hand side of the contour from point T_1 to a neighbouring point T_2 . The pivot S moves along an infinitesimally small arc S_1S_2 , in Figure 8 approximated by a straight-line segment. The arc radius of that sector equals the length of the polar arm PS .

We may regard the motion of tracing point T from T_1 to T_2 as a *rotation* of the tracing arm ST about the pivot point S_1 , followed by a *translation* of the tracing arm ST over the distance dh , *perpendicular* to ST .

This produces an imaginary auxiliary tracing point T_2' , which in this case lies outside the contour C , although it may also lie on or inside it.

The area of the region $PS_1T_1T_2S_2P$, that is, the area swept out by the articulated arm system PST , is, in *linear approximation*, equal to the sum of (1) the area of the sector PS_1S_2 , (2) the area of the sector $S_1T_1T_2'$, and (3) the area of the parallelogram $S_1T_2'T_2S_2$ ⁷.

⁷ Here lies the weakness of this explanation from [3], as well as of many similar variants. Because we make use of differentials, interpreted here as infinitesimal small quantities, various function differences are approximated linearly, and it is tacitly assumed that second- and higher-order terms of these differences are negligible compared to the first-order terms (Taylor expansion). Owing to this simplified line of reasoning, we assume, for example, that the planar region $T_1T_2'T_2T_1$ outside the contour C (see Figure 8) plays no role of importance. However, it remains questionable whether this assumption is actually justified.

We notice, (1) the area of the sector PS_1S_2 is $\frac{1}{2}PS^2 d\varphi$; (2) the area of the sector $S_1T_1T_2'$ is $\frac{1}{2}ST^2 d\sigma$; and (3) the area of the parallelogram $S_1T_2'T_2S_2$ is $ST dh$, so that, in linear approximation, the area of the infinitesimally small region $PS_1T_1T_2S_2P$ can be approximated by:

$$dA_R \approx \frac{1}{2}PS^2 d\varphi + \frac{1}{2}ST^2 d\sigma + ST dh \quad (1)$$

4.2. The measuring wheel

As with the original Amsler polar planimeter, in Figure 8 a small measuring wheel W is mounted to the left of the pivot point S , on the extension of tracing arm ST . The distance from the measuring wheel W to the pivot point S is denoted by w . As can be noticed in Figure 8 the length of w is constant, but the wheel position changes continuously while tracing point T traverses the contour of the region R . See the wheel positions W_1 , W_2 and W_2' .

While T tracing the segment of the contour from T_1 to T_2 , the motion of the measuring wheel W can be decomposed into a displacement from W_1 to W_2' , with approximate signed value $w \cdot (-d\sigma)$, followed by a displacement from W_2' to W_2 , with length dh .

If r is the radius of the measuring wheel W and α is the angle traversed by the rotating W , then it follows that:

$$r d\alpha = -w d\sigma + dh \rightarrow dh = w d\sigma + r d\alpha \quad (2)$$

So, the wheel's motion is decomposed into a *translational* and *rotational* contribution, leading to equation (2).

4.3. Amsler's formula

Combination of formulae (1) and (2), as well as integration yields the following equation (3):

$$A_R \propto \frac{1}{2}PS^2 \int d\varphi + \left(\frac{1}{2}ST^2 + wST \right) \int d\sigma + rST \int d\alpha \quad (3)$$

The polar arm PS doesn't make a complete revolution, so $\varphi_{\text{eind}} = \varphi_{\text{begin}}$ and $\int d\varphi = 0$. The same reasoning applies to the tracing arm ST : $\sigma_{\text{eind}} = \sigma_{\text{begin}}$, so $\int d\sigma = 0$. Hence, because neither arm completes a full rotation, the boundary terms of the integrals vanish, resulting in the remarkably simple formula:

$$A_R \propto rST\alpha \quad (4)$$

So, Jakob Amsler-Laffon essentially discovered that the area A_R of the region R is, to first order, directly proportional to the total angular displacement of the measuring wheel W , after the tracing of the contour C that encloses R .

Of course, an obvious objection is that this proof neglects all higher-order (Taylor-) terms.

5. A BRILLIANT INSIGHT: THE HIDDEN ROLE OF THE POLAR ARM

5.1. Sweeping the region R

It is noteworthy that formula (4) does not include the length of the polar arm PS, but only the length of the tracing arm ST⁸. Apparently, we could even leave the polar arm PS entirely out of consideration! This is also suggested in Figure 4 with Amsler's original drawing: there, the polar arm PS is drawn as an insignificant little line pointing downward, while the pole P is entirely absent. Yet the polar arm PS clearly performs an essential function.

Against the explanation in the previous section, mathematical objections can be raised (based on linear approximations only), but it has the advantage that it directly refers to the fundamental reasoning that Amsler himself also followed. See again his article [2], as well as the commentaries [1] and [4]. Apparently, those objections are not decisive, since the final result turns out to be correct.

Amsler's brilliant insight is that the area of a plane figure can be determined *exactly* by sweeping the region R, whose area A_R is to be determined, with a single line segment of *constant* length (in this case the tracing arm ST), which is what the extremely simple formula (4) expresses.

Anyone who studies the operation of predecessors of Amsler's polar planimeter, such as Zobel & Müller's (approximate) area-measuring machine, Hermann's conical planimeter (1814), Oppikofer's conical planimeter (1827), or Maxwell's design of a spherical planimeter (1855), see [6], all of which are mechanically and mathematically much more complicated than the polar planimeter, cannot help but be impressed by Amsler's brilliant insight. And one is left wondering how he came up with that idea.

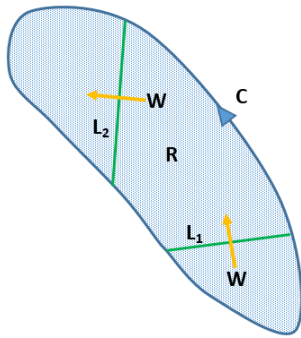
5.2. Straight-line segments

A revealing description of the idea implemented by Amsler in his polar planimeter can be found in [6, pp. 134–137]. Amsler came up with the idea of thinking of a region R in the plane as being composed of infinitely many straight-line segments. A subsequent line segment, at an infinitesimally small distance from a previously determined one, is always obtained via a rotation and a translation from that earlier line segment, as we likewise saw in a previous section. See also Figure 4 with Amsler's original Figure 2.

This leads to the problem that the region R can be thought of as being constructed from line segments in infinitely many different ways, and that these segments will moreover differ in length. See, for example, line segments L_1 and L_2 in Figure 9.

Amsler then had the idea that a tracing arm ST of *constant* length could be used as well, which is also reflected in formula (4). See Figure 10.

⁸ In the polar planimeter the length of the tracing arm ST can usually be adjusted. At the maximum length $ST = ST_{\max}$, the inaccuracy is approximately 0.1 cm^2 at scale 1:1, and therefore $0.1a^2$ at scale 1:a. If $ST = b \cdot ST_{\max}$, then it follows from formula (4) that the measuring wheel, for the same area A_R , indicates a value that is $1/b$ times as large; consequently, the inaccuracy becomes a factor b smaller.

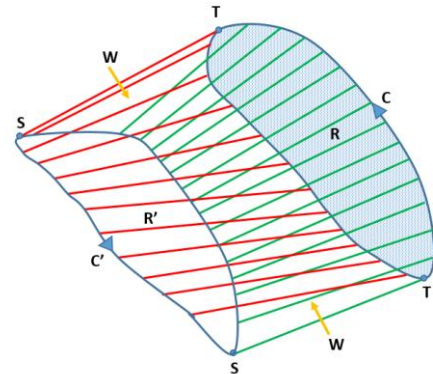
**Figure 9 (Left)**

A planar region R constructed from line segments. A measuring wheel W ensures a positive orientation when the contour C of the region R is traversed counter-clockwise. From line L_1 to line L_2 , the area measured by W therefore increases.

Figure 10 (Right)

The area swept out by the tracing arm ST (green from bottom to top (+); red from top to bottom (-)) as the tracing point T traverses the contour C once counterclockwise. In doing so, the

pivot point S traverses the contour C' of an imaginary region R' once. The yellow arrows indicate the direction of rotation of the measuring wheel, perpendicular to ST .



5.3. The pivot point S

Up to this point, we have focused on the tracing point T of ST , but let us now also consider the pivot point S , the other end of the tracing arm ST .

While in Figure 10 the tracing point T follows the right-hand side of the contour C , from bottom to top, the tracing arm ST (green) initially sweeps out region R , plus some region between R' and R , to the left of R . The orientation of the area swept out by ST is positive. The measuring wheel W registers an increase in area as the infinite number of green line segments ST with infinitesimally small mutual distances moves from bottom to top.

When the tracing arm ST reaches the top of contour C and the left side of the contour is subsequently traced, the tracing arm (now shown in red) moves downward and the orientation of the measuring wheel becomes negative. The measuring wheel W therefore registers a decrease in area because its direction of rotation has reversed. This ensures that the pivot point S forms a closed contour around an imaginary region R' during one tracing of the contour.

The area shaded in both green (+) and red (-) between the region to be measured, R , and the fictitious region R' is counted once positively and once negatively by the measuring wheel W . The positive and negative contributions cancel each other out, and therefore the doubly shaded area does not appear on the dial connected to W .

Because the lines passing through region R are shown in green (+), the area of region R is measured positively. Because the lines passing through the fictitious region R' are shown in red (-), the area of R' is measured negatively. Consequently, after contour C has been traced, the measuring wheel W indicate, apart from a scale factor, the area of R minus the area of R' .

5.4. Not a unique construction

It should be realized that the construction described above is not unique. There are infinitely many ways to construct Figure 10. For the given region R , neither the fictitious region R' nor its contour C' is uniquely determined. The shape of the fictitious region R' can therefore be chosen arbitrarily. Amsler came up with the brilliant idea of choosing the area of R' to be zero. He did this by taking R' , and therefore also its contour C' , to be a circular arc centred at the pole P with radius equal to the polar arm PS , and by restricting the pivot S to move only along that circular arc.

As a result, the shading shown in Figure 10 becomes uniquely defined, the area of R' equals zero, and the measuring wheel W indicates, apart from a scale factor, the exact area of R , after its boundary C has been traced once. Evidently, nothing more than the articulated arm system PST is required to determine the area of R exactly.

6. VECTORIAL INTERPRETATION OF THE POLAR PLANIMETER

6.1. An approach using Green's Theorem

Anyone seeking a mathematical explanation of the operating principle of Amsler's polar planimeter will find that, in most cases, the device is analysed using Green's theorem. The geometric argument presented above, based on geometry, is somewhat elaborate and tends to obscure Amsler's original intuitive insight, making it less immediately apparent why the instrument functions as it does.

By contrast, the vector-analytic treatment based on Green's theorem provides a considerably more elegant mathematical framework. However, this approach likewise fails to convey the intuitive geometric understanding of the planimeter's operation that appears to have motivated Amsler's original design.

In the Green's theorem approach, the analysis proceeds as follows. Let a Cartesian coordinate system and corresponding position vector $\vec{r}$ be introduced as shown in Figure 11.

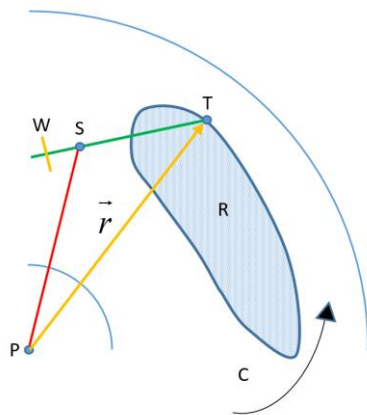


Figure 11

From the pole $P=(0,0)$ extends the polar arm PS , which is free to rotate about P . From the pivot point $S=(a,b)$ extends the tracing arm ST , which is free to rotate about S . The region R is enclosed by the curve C , which is traced in the counterclockwise direction by the tracing point $T=(x,y)$. The point T serves both as the endpoint of the tracing arm ST and as the endpoint of the radius vector originating from the pole P . A measuring wheel W (shown as a yellow line segment), is mounted perpendicular to the tracing arm ST ; in the figure, it is located to the left of the pivot point S .

The pole is chosen as $P=(0,0)$, so as the origin of a Cartesian coordinate system, typically chosen outside the region R , such that the entire region R is contained within the annular domain bounded by two concentric circles of radii $PS - ST$ and $PS + ST$, respectively.

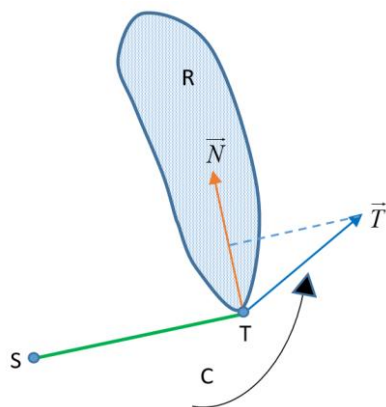


Figure 12

The unit tangent vector $\vec{T}$ is tangent to the curve C at the tracing point $T=(x,y)$. The unit normal vector $\vec{N}$ is perpendicular to the tracing arm ST at T . The projection of the tangent vector onto the normal vector determines the distance travelled by the measuring wheel W , as the curve C is traversed.

Figure 12 shows an arbitrarily chosen tracing point $T=(x,y)$ on the curve C . Also shown are the unit tangent vector $\vec{T}$ to contour C , at tracing point T , as well as the unit normal vector $\vec{N}$, perpendicular to the tracing arm ST .

6.2. Analysis of the measuring wheel

Because the measuring wheel W can rotate only in the direction perpendicular to the tracing arm ST , the displacement recorded by the wheel is determined by the projection of the tangent vector onto the normal vector, that is, by the inner product $\vec{N} \cdot \vec{T}$. See Figure 12.

If the tracing point T moves through an infinitesimal arc length ds , then the corresponding distance travelled by the measuring wheel W , in the direction perpendicular to ST , is given by:

$$\vec{N} \cdot \vec{T} ds = \vec{N} \cdot d\vec{r} \quad (5)$$

When the tracing point T completes a single traversal of the contour C , the measuring wheel W describes a closed curve C_W of length:

$$\oint_{C_W} \vec{N} \cdot \vec{T} ds = \oint_{C_W} \vec{N} \cdot d\vec{r} \quad (6)$$

In Figure 11, the measuring wheel W is located to the left of the pivot point S on the tracing arm ST extended in the leftward direction. However, it is essentially immaterial where along the tracing arm the measuring wheel is mounted.

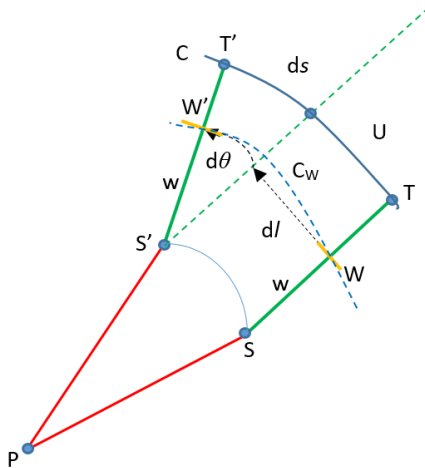


Figure 13

Two successive positions of the polar arm PS (red), the tracing arm ST (green), and the measuring wheel W (yellow), as the tracing point T , in positive counterclockwise orientation, slides along the curve from point T , via U , to neighbouring point T' , thereby traversing an infinitesimally small segment ds along the contour C . In doing so, the measuring wheel W undergoes a combined motion consisting of a translation dl , perpendicular to the tracing arm ST , resulting from the parallel displacement of the tracing arm ST to $S'U$, and a rotation through an angle $d\theta$ about the pivot point $S' \approx S$, arriving at the position W' .

If w denotes the distance between S and W , then the infinitesimal displacement ds in (6) can be decomposed into a *translational* component dl , perpendicular to the tracing arm ST , and a *rotational* component $w d\theta$, arising from the rotation of ST about the pivot point S . This idea has already been encountered above, but is illustrated more clearly in a slightly different manner in Figure 13. The left-hand integral in (6) can therefore be split into two integrals:

$$\oint_{C_W} \vec{N} \cdot \vec{T} ds = \oint_{C_W} dl + \oint_{C_W} w d\theta \quad (7)$$

Since the pole P lies outside the contour C , the cumulative angular change θ after a single traversal of C is equal to 0 radians. Consequently, the second integral on the right-hand side of (7) is equal to zero, so that:

$$\oint_{C_W} \vec{N} \cdot \vec{T} \, ds = \oint_{C_W} dl \quad (8)$$

Note: The distance w (from wheel W to pivot point S) does not appear in formula (8), and therefore the line integral is independent of the position of the measuring wheel W , on the tracing arm ST . The wheel W may even be placed (in the limiting case) at the tracing point T , in which case the measuring wheel W also traces the contour C . In that ultimate case:

$$\oint_{C_W} \vec{N} \cdot \vec{T} \, ds = \oint_C \vec{N} \cdot \vec{T} \, ds \quad (9)$$

7. A PROOF USING GREEN'S THEOREM

What relationship exists between the length $\oint_{C_W} \vec{N} \cdot \vec{T} \, ds = \oint_{C_W} \vec{N} \cdot d\vec{r}$ in (6) measured by the wheel

W and the area A_R of the region R, enclosed by contour C?



Figure 14

Green's mill in Sneiton, Nottingham. Source: https://en.wikipedia.org/wiki/Green%27s_Mill.

7.1. Some historical context

The physicist and mathematician *George Green* (1793–1841) was self-taught, the exceptionally gifted son of a miller who had received no more than a few years of primary education. While working in his father's grain mill, he succeeded in mastering the most advanced mathematics and physics, especially electrodynamics, of his time. See biography [14] and see [16].

No portraits of George Green are known to exist; however, his mill still stands today and now serves as a science centre. See Figure 14.



Figure 15

Mathematical physicist and telegraph engineer *Oliver Heaviside*, like *Green* a self-taught scholar, who reformulated electrodynamics in terms of vector analysis. See [12]. Heaviside reduced Maxwell's 20 equations to the 4 that we now call Maxwell's equations. The Heaviside (Kennelly) layer in the ionosphere is named after him.

Picture of Heaviside, circa 1900, IET Archive, Unknown author.

Green remained virtually unknown in the academic world until his fortieth year (1833). Most of his work remained unknown for even longer. Jakob Amsler-Laffon, who constructed his planimeter approximately 13 years after Green's early death, was likewise unaware of Green's work, which provides a vector-analytic explanation of the operation of his instrument.

It was only when scientists such as Hamilton, Gibbs, and

in particular *Oliver Heaviside* (1850–1925, see [12] and Figure 15), in the last quarter of the nineteenth century, began to reformulate Faraday's and Maxwell's electrodynamics in vector-analytic terms, that they discovered that Green had, decades earlier, already formulated a large number of fundamental concepts and their interrelations in mathematical physics (including, for example, the concept of potential), without this having been known at the time. See [15, pp. 49 ff. and pp. 83 ff.].

Figure 16

James Clerk Maxwell, who formulated the laws of electromagnetism, whose status is comparable to that of Newton's laws in mechanics. The formulation of these laws was greatly simplified by the vector-analytic description by Oliver Heaviside, who made use of the almost unknown work of George Green. Maxwell also designed a planimeter. Portrait after Fergus of Greenock. Gravure of engraving by Stodart based on photograph by Fergus.



In this context, it is also interesting to know that Maxwell, in the years 1854/1855, had devised a complicated spherical planimeter which probably would not have worked very well due to expected mechanical problems, but he immediately abandoned it when he first saw Amsler's strikingly simpler polar planimeter. See [15, p. 20].

7.2. What is Green's theorem?

Green's theorem in the plane is a special case of Stokes' theorem, one of the principal theorems of vector analysis, which establishes a relationship between the concepts of *circulation* on a contour and the *cumulative rotation* (curl) of a vector field on the enclosed region. The particular form of Green's theorem referred to here is stated as follows (in the formulation given by Augustin-Louis Cauchy in 1846 for the theory of complex functions):

$$\oint_C Fdx + Gdy = \iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy \quad (10)^9$$

Formula (10) establishes a relationship between the simple closed contour C and the region R enclosed by C . In this formula F and G are the entries of a planar vector field $\vec{v}(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$ on the region R in the (x, y) -plane.

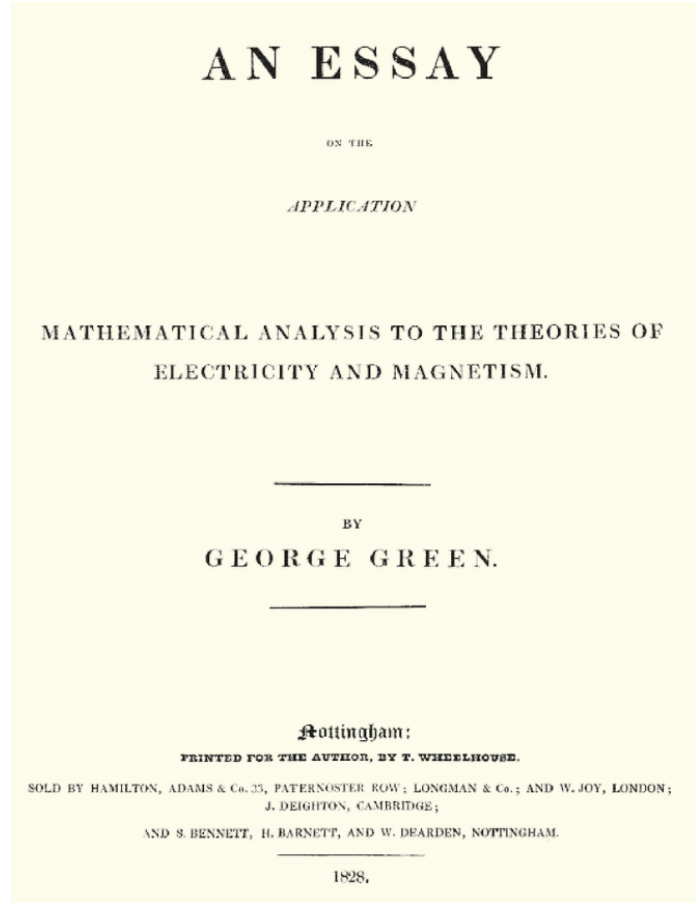
⁹ Remarkably, this well-known formula (10) does not appear in George Green's 1828 paper. What later came to be known as Green's theorem in the plane was not formulated by Green himself. Green was interested in developing a unified formulation of electromagnetism, which at the time was a rather disordered collection of empirical relationships discovered by Hans Christian Ørsted, Joseph Henry, Michael Faraday, and André-Marie Ampère between electrical and magnetic phenomena that were still regarded as largely separate. Green discovered and proved a theorem of which formula (10), as was later recognized, is a special case, and which is itself also a special case of Stokes' theorem. See [14], [10, pp. 120 ff.], and [16, pp. 100 ff.].

The integral $\oint_C Fdx + Gdy = \oint_C \vec{v} \cdot d\vec{r}$ is the *circulation* of the vector field along the curve C. The difference between the partial derivatives of F and G (the z-component of a cross product) represents the local rotation (curl) in a specific point of the field in the region R, and therefore the double integral $\iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy$ is the *cumulative rotation* of the vector field over the total of region R. See [11, pp. 126 ff.].

Figure 17

The paper privately published by George Green in 1828, containing the much more general theorem of which what is now known as Green's theorem in the plane was later recognized to be a special case. See [14]. The paper was read by 51 people, 50 of whom were friends and acquaintances who kindly purchased a copy but presumably understood little or nothing of it, and one mathematician who also did not fully understand it, yet recognized Green's mathematical genius and, a few years later, arranged for Green to study at the University of Cambridge at the age of 40 (sic). Unfortunately, Green died only a few years later.

Source: [Wikipedia article on George Green](#).



Variants of the proof, using Green's theorem in the plane, that Amsler's polar planimeter indeed measures the area of the region R exactly, up to a constant scale factor, can be found, for example, in [7] and [19]. A very comprehensive analysis of various mathematical aspects of planimeters can be found in [10].

7.3. Proof of the Amsler planimeter by Green's theorem

The pivot point is $S = (a, b)$; the tracing point is $T = (x, y)$. For the normal vector $\vec{N}$, with length 1, perpendicular to the tracing arm ST, see Figure 12, applies:

$$\vec{N} = \frac{1}{ST} \begin{pmatrix} b-y \\ x-a \end{pmatrix} \quad (11)$$

Hence:

$$\oint_C \vec{N} \cdot \vec{T} \, ds = \oint_C \vec{N} \cdot d\vec{r} = \frac{1}{ST} \oint_C (b-y)dx + (x-a)dy \quad (12)$$

In the right integral $\oint_C (b-y)dx + (x-a)dy$ is apparently: $F(x, y) = b-y$ and $G(x, y) = x-a$ (13)

From this we find the partial derivatives:

$$\frac{\partial G}{\partial x} = 1 - \frac{\partial a}{\partial x} \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{\partial b}{\partial y} - 1 \quad (14)$$

And therefore, the local rotation (curl) of the vector field formed by F and G $\vec{v}(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$ equals:

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 2 - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \right) \quad (15)$$

From Figure 3 follow the next equations: $a^2 + b^2 = PS^2$ and $(x - a)^2 + (y - b)^2 = ST^2$.

Differentiating these two equations yields the following two pairs of equations, each with two partial derivatives as unknowns:

$$\begin{cases} a \frac{\partial a}{\partial x} + b \frac{\partial b}{\partial x} = 0 & (a) \\ a \frac{\partial a}{\partial y} + b \frac{\partial b}{\partial y} = 0 & (b) \end{cases} \quad \text{and} \quad \begin{cases} x \frac{\partial a}{\partial x} + y \frac{\partial b}{\partial x} = x - a & (c) \\ x \frac{\partial a}{\partial y} + y \frac{\partial b}{\partial y} = y - b & (d) \end{cases} \quad (16)$$

From equations (a) and (c), it follows for the derivatives with respect to x :

$$\begin{cases} \frac{\partial a}{\partial x} = -\frac{b(x-a)}{\Delta} \\ \frac{\partial b}{\partial x} = \frac{a(x-a)}{\Delta} \end{cases} \quad (17)$$

From equations (b) and (d), it follows for the derivatives with respect to y :

$$\begin{cases} \frac{\partial a}{\partial y} = -\frac{b(y-b)}{\Delta} \\ \frac{\partial b}{\partial y} = \frac{a(y-b)}{\Delta} \end{cases} \quad (18)$$

Where the denominators of (17) and (18) are equal to:

$$\Delta = ay - bx \quad (19)$$

Hence:

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = \frac{\Delta}{\Delta} = 1 \quad (20)$$

Thus, according to (15), the curl of the vector field $\vec{v}(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$ defined by F and G equal to 1 in every single point of the region R :

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 2 - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \right) = 1 \quad (21)$$

The total circulation (or cumulative curl) of that vector field over the region R is therefore equal to the area of R .

After all:

$$\iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy = \iint_R 1 dx dy = A_R \quad (22)$$

In summary, it follows that the distance travelled by the measuring wheel W perpendicular to the tracing arm is proportional to:

$$\oint_{C_W} \vec{N} \cdot \vec{T} ds \propto \oint_C \vec{N} \cdot \vec{T} ds = \oint_C \vec{N} \cdot d\vec{r} = \frac{1}{ST} \iint_R 1 dx dy \propto A_R \quad (23)$$

8. CONCLUSIONS

8.1. Amsler's geometrical explanation

Jakob Amsler-Laffon derived the formula for the polar planimeter by analysing the geometrical kinematics of the device. The fundamental principle is that the measuring wheel records only the component of motion *perpendicular* to its axis of rotation. This corresponds, in modern terms, to an inner product formulation of the measurement process.

During tracing of a closed curve, the instrument undergoes a combined translational and rotational motion. Amsler established that the total angular displacement of the measuring wheel is proportional to the enclosed area. By decomposing infinitesimal displacements and integrating over one complete traversal of the boundary, he obtained:

$$A_R \propto rST\alpha$$

where:

- A_R is the area of region R enclosed by contour C
- ST is the length of the tracing arm of the planimeter
- α is the total measured wheel displacement in radians.

8.2. Explanation by Green's theorem

A rigorous justification of the working of the Amsler's polar planimeter follows from vector calculus, especially Green's theorem. However, Amsler did not formulate his result in these terms, most probably he hadn't any knowledge of that theorem.

The mathematical basis of this result can be expressed in terms of line integrals, where the area of a planar region is computed via an integral along its boundary. The polar planimeter provides a mechanical realization of this integral, yielding the enclosed area up to a constant scaling factor.

9. ACKNOWLEDGEMENTS

My thanks go to John Vossepel, a well-known enthusiastic collector of planimeters, who sent me the internet references [1], [2], [4], [17] to [23], as well as the book [6]. For me he was a valuable source of more recent historical information on planimeters.

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11. APPENDIX: VECTOR ANALYSIS CONCEPTS USED

See for instance [11] chapters 3 to 6

11.1. Rotation (curl) of a vector field

The curl of a vector field $\vec{F}(x, y, z) = \begin{pmatrix} F_x(x, y, z) \\ F_y(x, y, z) \\ F_z(x, y, z) \end{pmatrix}$ is defined as:

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \mathbf{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

with the nabla-operator $\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$

Curl consists of three components:

curl about the x-axis; curl about the y-axis; curl about the z-axis

The curl is a vector that indicates the strength and axis of the *local* rotation of a vector field.

Example

$$\vec{F}(x, y, z) = \begin{pmatrix} r \cos(\omega t) \\ r \sin(\omega t) \\ 0 \end{pmatrix}; \quad r = \sqrt{x^2 + y^2} \quad]; \text{ hence: } F_x = r \cos(\omega t), \quad F_y = r \sin(\omega t), \quad F_z = 0.$$

No dependence on z : $F_z = 0$, so $z: \frac{\partial F_y}{\partial z} = 0 \quad \frac{\partial F_x}{\partial z} = 0$ and $\nabla \times \vec{F} = \mathbf{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$

$$\frac{\partial F_y}{\partial x} = \sin(\omega t) \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial F_x}{\partial y} = \cos(\omega t) \frac{y}{\sqrt{x^2 + y^2}}, \quad \text{hence } \nabla \times \vec{F} = \frac{x \sin(\omega t) - y \cos(\omega t)}{\sqrt{x^2 + y^2}} \mathbf{k}$$

Interpretation

Here, the curl points entirely in the z -direction, with unit vector $\mathbf{k}$, so, the vector field rotates in the (x, y) -plane. Mind, the curl's magnitude depends on position and time.

Unit of Curl

The unit of the curl depends on the unit of the vector field itself. From the definition of the curl, it

follows that $\nabla \times \vec{F}$ in $\frac{\text{unit of } \vec{F}}{\text{length}}$.

For instance, electromagnetism: for an electric field: $\vec{E}$ in $\text{V/m} \rightarrow \nabla \times \vec{E}$ in V/m^2 .

Important physical insight, the curl measures a change per unit distance, or a circulation per unit area. Therefore, a factor of inverse length always appears in its unit.

11.2. Stokes' theorem

For a vector field the following holds: $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$.

Stokes' theorem holds provided that:

- S is an orientable smooth (or piecewise smooth) surface;
- ∂S is a closed piecewise smooth boundary curve;
- $\vec{F}$ is continuously differentiable on an open neighbourhood of S ;
- the orientations of the surface and its boundary are chosen consistently.

The requirement that the vector field be smooth on an open neighbourhood of the surface is particularly important and is often overlooked.

Interpretation

Stokes' theorem states that the total circulation of a vector field around a closed curve equals the total curl passing through any surface bounded by that curve. The line integral around the boundary measures the field's circulation. The surface integral of the curl measures the *accumulated* local rotation over the surface. Stokes' theorem shows that these two quantities are exactly equal.

Physical Meaning

In fluid dynamics the line integral represents the circulation of the fluid around a closed loop. The curl represents the *local* rotational motion (vorticity) of the fluid. Stokes' theorem connects the microscopic rotation within the surface to the macroscopic circulation along its boundary.

Conditions

Stokes' theorem applies when: the vector field has continuous partial derivatives on and near the surface; the surface is a smooth, orientable surface; the orientation of boundary and surface follows the *right-hand rule*, well known from the theory of electrodynamics.

A common pitfall

Given the following vector field:

$$\vec{F}(x, y, z) = \begin{bmatrix} -\frac{y}{x^2 + y^2} \\ \frac{x}{x^2 + y^2} \\ 0 \end{bmatrix}.$$

We have three component functions:

$$P = \frac{-y}{x^2 + y^2}, Q = \frac{x}{x^2 + y^2}, R = 0.$$

Since P , Q and R do not depend on z , and the first two components of the curl are zero.

We therefore only need to compute:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Using the quotient rule, we find:

$$\frac{\partial Q}{\partial x} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial P}{\partial y} = \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Hence:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Therefore, the curl equals the zero-vector, everywhere on the plane, except when $(x,y) = (0,0)$:

$$\nabla \times \vec{F} = \vec{0}$$

So, for all points where $x^2 + y^2 \neq 0$, that is, everywhere except on the z-axis.

It is important to note that the vector field is not defined on the z-axis, so the curl does not exist there. This is precisely why one cannot directly apply Stokes' theorem (or Green's theorem in two dimensions, see hereafter) when the chosen surface encloses the z-axis (or, equivalently in 2D, the origin).

11.3. Green's theorem

Green's theorem (of the plane) is a special case of Stokes' theorem.

For a vector field $\vec{F}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ in the (x,y) -plane: $\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$.

Where C is a closed curve bounding the region R, and the curve is traversed *counterclockwise*.

Green's theorem is the two-dimensional version of Stokes' theorem. The crucial condition in both cases is that the vector field be continuously differentiable on an open neighbourhood of the entire region (or surface), not merely on the boundary itself.

How Green's theorem follows from Stokes' theorem

Consider a flat surface S lying in the (x,y) -plane and define the 3D vector field:

$$\vec{F}(x, y, z) = \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix}, \text{ then the curl of this vector field is: } \nabla \times \vec{F} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}.$$

The unit base vector $\mathbf{k}$ is the unit normal to the (x, y) -plane.

$$\text{Therefore, the inner product yields: } (\nabla \times \vec{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Substituting this into Stokes' theorem gives $\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$, which is exactly Green's theorem.

Interpretation

Green's theorem relates the circulation around a closed curve in a plane to the total rotation inside the enclosed region. Stokes' theorem extends the same idea to an arbitrary surface in 3D.

Area calculation using Green's theorem

Using Green's theorem, the area of a plane region R can be computed from a line integral taken along its boundary C .

$$P = -\frac{y}{2}; Q = \frac{x}{2}; \frac{\partial Q}{\partial x} = \frac{1}{2}; \frac{\partial P}{\partial y} = -\frac{1}{2} \rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Thus, the right double integral in Green's formula equals the area A_R of region R .

Hence:

$$A_R = \frac{1}{2} \oint_C (x dy - y dx)$$

Area in polar coordinates

$$x = r \cos \theta, y = r \sin \theta.$$

So:

$$\begin{aligned} dx &= \cos \theta dr - r \sin \theta d\theta, \\ dy &= \sin \theta dr + r \cos \theta d\theta. \end{aligned}$$

Substituting:

$$\begin{aligned} x dy - y dx &= r \cos \theta (\sin \theta dr + r \cos \theta d\theta) - r \sin \theta (\cos \theta dr - r \sin \theta d\theta). \end{aligned}$$

We recognize:

$$r \cos \theta \sin \theta dr - r \sin \theta \cos \theta dr = 0.$$

As a result, we find:

$$r^2 (\cos^2 \theta + \sin^2 \theta) d\theta = r^2 d\theta.$$

Hence:

$$A_R = \frac{1}{2} \oint r^2 d\theta.$$

Example: A circle with radius R

Parameterize the circle by: $x = R \cos t, y = R \sin t, 0 \leq t \leq 2\pi.$

Then: $dx = -R \sin t dt, dy = R \cos t dt.$

Substituting yields: $x dy - y dx = R^2 (\cos^2 t + \sin^2 t) dt = R^2 dt.$

Hence:

$$A_R = \frac{1}{2} \int_0^{2\pi} R^2 dt = \frac{1}{2} (2\pi R^2) = \pi R^2$$

as expected.

11.4. Relation to the Amsler planimeter

The famous Jakob Amsler-Laffon planimeter is based precisely on this idea. The instrument mechanically evaluates a line integral along the boundary of a region and directly produces its area. In modern mathematical language, its operation can be elegantly explained using Green's theorem. For this reason, the Amsler planimeter is often cited as a classic practical application of Green's theorem.